March 31th, 2005

Characteristic initial value problem and its application in quasi-local energy expression

Xiaoning Wu

Department of Physics, National Central University,
Chung-li 32054, Taiwan
e-mail : wuxn@phy.ncu.edu.tw

1 A brief introduction of Newman-Penrose formulism

We know that null geodesic is characteristic curve in classical general relativity, espcielly for vacuum case. The behavior of null geodesics play an impotent rule when we try to understand the properties of the space-time. The main idea of Newman-Penrose formulism is just trying to catch the behavior of the two null normals of a space-like 2-sphere in the space-time and to understand the properties of space-time. Unlike the standard cauchy problem of Einstein equations, we will start from two intersected light cones N_1 and N_2 instead of a standard cauchy surface Σ.

In a general space-time (M, g_{ab}), we can always find a set of comlex null tetrad $\{l,\ n,\ m,\ \bar{m}\}$, where l and n are two real null vectors and m, $\bar{m}$ are two complex null vectors. The normalization conditions are

$$l \cdot n = -1 \qquad and \qquad m \cdot \bar{m} = 1. \tag{1}$$

Suppose we have a space-like two surface S, where X and Y are the basis of the tangent space of S, we can choose l and n are the two null normals and $m = X + iY$. Here we find there are some freedom in the choices of the null tetrad. The first one is we can make a phase transformation on m, i.e. $m \to e^{i\omega} m$, where ω is a function on S. Second is we can change the " length" of l and n, i.e. $l \to fl$ and $n \to \frac{1}{f} n$, where f is also a function on S. In fact, the first one corresponds to the rotation transformation and second corresponds to boost transformation.

Given the null tetrad, we denote $D := l^a \nabla_a$, $D' := n^a \nabla_a$, $\delta := m^a \nabla_a$ and $\bar{\delta} := \bar{m}^a \nabla_a$, then we can define the complex connection coefficients as [1, 2, 3, 4]

$$\begin{array}{ccccc} & m^a \nabla l_a & n^a \nabla l_a & \bar{m}^a \nabla m_a & \bar{m}^a \nabla n_a \\ D & -\kappa & -(\varepsilon + \bar{\varepsilon}) & \varepsilon - \bar{\varepsilon} & \pi \\ D' & -\tau & -(\gamma + \bar{\gamma}) & \gamma - \bar{\gamma} & \nu \\ \delta & -\sigma & -(\bar{\alpha} + \beta) & -\bar{\alpha} + \beta & \mu \\ \bar{\delta} & -\rho & -(\alpha + \bar{\beta}) & \alpha - \bar{\beta} & \lambda \end{array} . \tag{2}$$

We call these 12 complex connection coefficients as Newman-Penrose coefficients because they are defined by E.T.Newman and R.Penrose first[4]. Some of these coefficients have very good geometric meaning. For example, $\kappa = 0$ tells us l is the tangent vector of null geodesic. If l is a geodesic, $(\varepsilon + \bar{\varepsilon}) = 0$ tells us this geodesic is affine parameterized. The real part of ρ is the expansion of l and the image part of it is 2-dim twist of l. σ is the shear of l. The image part of ε and π tell us whether the tetrad is parallelly transported along l. Another advantage of N-P coefficients is they are symmetric for l and n, i.e. $(\nu,\gamma,\mu,\lambda,\tau)$ has similar meaning for n as $(\kappa,\varepsilon,\rho,\sigma,\pi)$ for l. $(\alpha - \bar{\beta})$ is the 2-dim connection coefficient.

Furthurmore, we know the irreducible repression of Riemann curvature are [1]

$$R_{abcd} = C_{abcd} + E_{abcd} + G_{abcd}, \tag{3}$$

where

$$\begin{aligned} E_{abcd} &= \frac{1}{2}(g_{ac}S_{bd} + g_{bd}S_{ac} - g_{ad}S_{bc} - g_{bc}S_{ad}), \\ G_{abcd} &= \frac{R}{12}(g_{ac}g_{bd} - g_{ad}g_{bc}), \\ S_{ab} &= R_{ab} - \frac{R}{4}g_{ab}. \end{aligned}$$

We can define the Newman-Penrose components of above irreducible part as [1, 2, 3]

$$\begin{aligned}
\Psi_0 &:= C_{abcd}l^a m^b l^c m^d, \\
\Psi_1 &:= C_{abcd}l^a n^b l^c m^d, \\
\Psi_2 &:= \frac{1}{2}\, C_{abcd}l^a n^b (l^c n^d - m^c \bar{m}^d), \\
\Psi_3 &:= C_{abcd}n^a l^b n^c \bar{m}^d, \\
\Psi_4 &:= C_{abcd}n^a \bar{m}^b n^c \bar{m}^d, \\
\Phi_{00} &:= \frac{1}{2} S_{ab}l^a l^b, \\
\Phi_{01} &:= \frac{1}{2} S_{ab}l^a m^b, \\
\Phi_{02} &:= \frac{1}{2} S_{ab}m^a m^b, \\
\Phi_{11} &:= \frac{1}{4} S_{ab}(l^a n^b + m^a \bar{m}^b), \\
\Phi_{12} &:= \frac{1}{2} S_{ab}n^a m^b, \\
\Phi_{22} &:= \frac{1}{2} S_{ab}l^a l^b, \\
\Lambda &:= \frac{R}{24}.
\end{aligned} \tag{4}$$

With the tetrad and connection, the torsion-free condition, the standard Carton structure equations and Bianchi identities have following form [1, 2, 3, 4]:

1. The Newman-Penrose equations (structure equations) are

$$\begin{aligned}
D\rho - \bar{\delta}\kappa &= \rho^2 + |\sigma|^2 + (\varepsilon + \bar{\varepsilon})\rho - \bar{\kappa}\tau - \kappa(3\alpha + \bar{\beta} - \pi) + \Phi_{00}, && (5)\\
D\sigma - \delta\pi &= (\rho + \bar{\rho})\sigma + (3\varepsilon - \bar{\varepsilon})\sigma - (\tau - \bar{\pi} + \bar{\alpha} + 3\beta)\kappa + \Psi_0, && (6)\\
D\tau - D'\kappa &= (\tau + \bar{\pi})\rho + (\bar{\tau} + \pi)\sigma + (\varepsilon - \bar{\varepsilon})\tau - (3\gamma + \bar{\gamma})\kappa + \Psi_1 + \Phi_{01}, && (7)\\
D\alpha - \bar{\delta}\varepsilon &= (\rho + \bar{\varepsilon} - 2\varepsilon)\alpha + \beta\bar{\sigma} - \bar{\beta}\varepsilon - \kappa\lambda - \bar{\kappa}\gamma + (\varepsilon + \rho)\pi + \Phi_{10}, && (8)\\
D\beta - \delta\varepsilon &= (\alpha + \pi)\sigma + (\bar{\rho} - \bar{\varepsilon})\beta - (\mu + \gamma)\kappa - (\bar{\alpha} - \bar{\pi})\varepsilon + \Psi_1, && (9)\\
D\gamma - D'\varepsilon &= (\tau + \bar{\pi})\alpha + (\bar{\tau} + \pi)\beta - (\varepsilon + \bar{\varepsilon})\gamma - (\gamma + \bar{\gamma})\varepsilon \\
&\quad + \tau\pi - \nu\kappa + \Psi_2 + \Phi_{11} - \Lambda, && (10)\\
D\lambda - \bar{\delta}\pi &= (\rho\lambda + \bar{\sigma}\mu) + \pi^2 + (\alpha - \bar{\beta})\pi - \nu\bar{\kappa} - (3\varepsilon - \bar{\varepsilon})\lambda + \Phi_{20}, && (11)\\
D\mu - \delta\pi &= (\rho\lambda + \bar{\sigma}\lambda) + |\pi|^2 - (\varepsilon + \bar{\varepsilon})\mu - \pi(\bar{\alpha} - \beta) - \nu\kappa + \Psi_2 + 2\Lambda, && (12)\\
D\nu - D'\pi &= (\pi + \bar{\tau})\mu + (\bar{\pi} + \tau)\lambda + (\gamma - \bar{\gamma})\pi - (3\varepsilon + \bar{\varepsilon})\nu + \Psi_3 + \Phi_{21}, && (13)\\
D'\lambda - \bar{\delta}\nu &= -(\mu + \bar{\mu})\lambda - (3\gamma - \bar{\gamma})\lambda + (3\alpha + \bar{\beta} + \pi - \bar{\tau})\nu - \Psi_4, && (14)\\
\delta\rho - \bar{\delta}\sigma &= \rho(\bar{\alpha} + \beta) - \sigma(2\alpha - \bar{\beta}) + (\rho - \bar{\rho})\tau + (\mu - \bar{\mu})\kappa - \Psi_1 + \Phi_{01}, && (15)\\
\delta\alpha - \bar{\delta}\beta &= (\mu\rho - \lambda\sigma) + |\alpha|^2 + |\beta|^2 - 2\alpha\beta + \gamma(\rho - \bar{\rho}) + \varepsilon(\mu - \bar{\mu}) \\
&\quad - \Psi_2 + \Phi_{11} + \Lambda, && (16)\\
\delta\lambda - \bar{\delta}\mu &= (\rho - \bar{\rho})\nu + (\mu - \bar{\mu})\pi + \mu(\alpha + \bar{\beta}) + \lambda(\bar{\alpha} - 3\beta) - \Psi_3 + \Phi_{21}, && (17)\\
\delta\nu - D'\mu &= (\mu^2 + |\lambda|^2) + (\gamma + \bar{\gamma})\mu - \bar{\nu}\pi + (\tau - 3\beta - \bar{\alpha})\nu + \Phi_{22}, && (18)\\
\delta\gamma - D'\beta &= (\tau - \bar{\alpha} - \beta)\gamma + \mu\tau - \sigma\nu - \varepsilon\bar{\nu} - \beta(\gamma - \bar{\gamma} - \mu) + \alpha\bar{\lambda} + \Phi_{12}, && (19)\\
\delta\tau - D'\sigma &= (\mu\sigma + \bar{\lambda}\rho) + (\tau + \beta - \bar{\alpha})\tau - (3\gamma - \bar{\gamma})\sigma - \kappa\bar{\nu} + \Phi_{02}, && (20)
\end{aligned}$$

$$\begin{aligned}
D'\rho - \bar{\delta}\tau &= -(\rho\bar{\mu} + \sigma\lambda) + (\bar{\beta} - \alpha - \bar{\tau})\tau + (\gamma + \bar{\gamma})\rho + \nu\kappa - \Psi_2 - 2\Lambda, && (21)\\
D'\alpha - \bar{\delta}\gamma &= (\rho + \varepsilon)\nu - (\tau + \beta)\lambda + (\bar{\gamma} - \bar{\mu})\alpha + (\bar{\beta} - \bar{\tau})\gamma - \Psi_3. && (22)
\end{aligned}$$

2. Commutators (Torsion-free condition)

$$\begin{aligned}
D'D - DD' &= (\gamma + \bar{\gamma})D + (\varepsilon + \bar{\varepsilon})D' - (\tau + \bar{\pi})\bar{\delta} - (\bar{\tau} + \pi)\delta, && (23)\\
\delta D - D\delta &= (\bar{\alpha} + \beta - \bar{\pi})D + \kappa D' - \sigma\bar{\delta} - (\bar{\rho} + \varepsilon - \bar{\varepsilon})\delta, && (24)\\
\delta D' - D'\delta &= -\bar{\nu}D + (\tau - \bar{\alpha} - \beta)D' + \bar{\lambda}\bar{\delta} + (\mu - \gamma + \bar{\gamma})\delta, && (25)\\
\bar{\delta}\delta - \delta\bar{\delta} &= (\bar{\mu} - \mu)D + (\bar{\rho} - \rho)D' - (\bar{\alpha} - \beta)\bar{\delta} - (\bar{\beta} - \alpha)\delta. && (26)
\end{aligned}$$

3. Bianchi identities

$$\begin{aligned}
\bar{\delta}\Psi_0 - D\Psi_1 + D\Phi_{01} - \delta\Phi_{00} &= (4\alpha - \pi)\Psi_0 - 2(2\rho + \varepsilon)\Psi_1 + 3\kappa\Psi_2\\
&\quad +(\bar{\pi} - 2\bar{\alpha} - 2\beta)\Phi_{00} + 2(\varepsilon + \bar{\rho})\Phi_{01}\\
&\quad +2\sigma\Phi_{10} - 2\kappa\Phi_{11} - \bar{\kappa}\Phi_{02}, && (27)\\
D'\Psi_0 - \delta\Psi_1 + D\Phi_{02} - \delta\Phi_{01} &= (4\gamma - \mu)\Psi_0 - 2(2\tau + \beta)\Psi_1 + 3\sigma\Psi_2\\
&\quad +(2\varepsilon - 2\bar{\varepsilon} + \bar{\rho})\Phi_{02} + 2(\bar{\pi} - \beta)\Phi_{01}\\
&\quad +2\sigma\Phi_{11} - 2\kappa\Phi_{12} - \bar{\lambda}\Phi_{00}, && (28)\\
\bar{\delta}\Psi_3 - D\Psi_4 + \bar{\delta}\Phi_{21} - D'\Phi_{20} &= (4\varepsilon - \rho)\Psi_4 - 2(2\pi + \alpha)\Psi_3 + 3\lambda\Psi_2\\
&\quad +(2\gamma - 2\bar{\gamma} + \bar{\mu})\Phi_{20} + 2(\bar{\tau} - \alpha)\Phi_{21}\\
&\quad +2\lambda\Phi_{11} - 2\nu\Phi_{10} - \bar{\sigma}\Phi_{22}, && (29)\\
D'\Psi_3 - \delta\Psi_4 + \bar{\delta}\Phi_{22} - D'\Phi_{21} &= (4\beta - \tau)\Psi_4 - 2(2\mu + \gamma)\Psi_3 + 3\nu\Psi_2\\
&\quad +(\bar{\tau} - 2\bar{\beta} - 2\alpha)\Phi_{22} + 2(\gamma + \bar{\mu})\Phi_{21}\\
&\quad +2\lambda\Phi_{12} - 2\nu\Phi_{11} - \bar{\nu}\Phi_{20}, && (30)\\
D\Psi_2 - \bar{\delta}\Psi_1 + D'\Phi_{00} - \bar{\delta}\Phi_{01} + 2D\Lambda &= -\lambda\Psi_0 + 2(\pi - \alpha)\Psi_1 + 3\rho\Psi_2 - 2\kappa\Psi_3\\
&\quad +(2\gamma + 2\bar{\gamma} - \bar{\mu})\Phi_{00} - 2(\bar{\tau} + \alpha)\Phi_{01}\\
&\quad -2\tau\Phi_{10} + 2\rho\Phi_{11} + \bar{\sigma}\Phi_{02}, && (31)\\
D'\Psi_2 - \delta\Psi_3 + D\Phi_{22} - \delta\Phi_{21} + 2D'\Lambda &= \sigma\Psi_4 + 2(\beta - \tau)\Psi_3 - 3\mu\Psi_2 + 2\nu\Psi_1\\
&\quad +(\bar{\rho} - 2\varepsilon - 2\bar{\varepsilon})\Phi_{22} + 2(\bar{\pi} + \beta)\Phi_{21}\\
&\quad +2\pi\Phi_{12} - 2\mu\Phi_{11} - \bar{\lambda}\Phi_{20}, && (32)\\
D\Psi_3 - \bar{\delta}\Psi_2 - D\Phi_{21} + \delta\Phi_{20} - 2\bar{\delta}\Lambda &= -\kappa\Psi_4 + 2(\rho - \varepsilon)\Psi_3 + 3\pi\Psi_2 - 2\lambda\Psi_1\\
&\quad +(2\bar{\alpha} - 2\beta - \bar{\pi})\Phi_{20} - 2(\bar{\rho} - \varepsilon)\Phi_{21}\\
&\quad -2\pi\Phi_{11} + 2\mu\Phi_{10} + \bar{\kappa}\Phi_{22}, && (33)\\
D'\Psi_1 - \delta\Psi_2 - D'\Phi_{01} + \bar{\delta}\Phi_{02} - 2\delta\Lambda &= \nu\Psi_0 + 2(\gamma - \mu)\Psi_1 - 3\tau\Psi_2 + 2\sigma\Psi_3\\
&\quad +(\bar{\tau} - 2\bar{\beta} + 2\alpha)\Phi_{02} + 2(\bar{\mu} - \gamma)\Phi_{01}\\
&\quad +2\tau\Phi_{11} - 2\rho\Phi_{12} - \bar{\nu}\Phi_{00}, && (34)\\
D\Phi_{11} - \delta\Phi_{10} - \bar{\delta}\Phi_{01} + D'\Phi_{00} + 3D\Lambda &= (2\gamma - \mu + 2\bar{\gamma} - \bar{\mu})\Phi_{00} + (\pi - 2\alpha - 2\bar{\tau})\Phi_{01}\\
&\quad +(\bar{\pi} - 2\bar{\alpha} - 2\tau)\Phi_{10} + 2(\rho + \bar{\rho})\Phi_{11}\\
&\quad +\bar{\sigma}\Phi_{02} + \sigma\Phi_{20} - \bar{\kappa}\Phi_{12} - \kappa\Phi_{21}, && (35)\\
D\Phi_{12} - \delta\Phi_{11} - \bar{\delta}\Phi_{02} + D'\Phi_{01} + 3\delta\Lambda &= (-2\alpha + 2\bar{\beta} + \pi - \bar{\tau})\Phi_{02} + (\bar{\rho} + 2\rho - 2\bar{\varepsilon})\Phi_{12}\\
&\quad +2(\bar{\pi} - \tau)\Phi_{11} + (2\gamma - 2\bar{\mu} - \mu)\Phi_{01}\\
&\quad +\bar{\nu}\Phi_{00} - \bar{\lambda}\Phi_{10} + \sigma\Phi_{21} - \kappa\Phi_{22}, && (36)
\end{aligned}$$

$$
\begin{aligned}
D\Phi_{22} - \delta\Phi_{21} - \bar{\delta}\Phi_{12} + D'\Phi_{11} + 3D'\Lambda \;\; = \;\; & (\rho + \bar{\rho} - 2\varepsilon - 2\bar{\varepsilon})\Phi_{22} + (2\bar{\beta} + 2\pi - \bar{\tau})\Phi_{12} \\
& + (2\beta + 2\bar{\pi} - \tau)\Phi_{21} - 2(\mu + \bar{\mu})\Phi_{11} \\
& + \nu\Phi_{01} + \bar{\nu}\Phi_{10} - \bar{\lambda}\Phi_{20} - \lambda\Phi_{02}. \qquad (37)
\end{aligned}
$$

Above 33 equations are what we want to deal with. The first sight of these equations are hopeless because they are so complex and strongly coupled. What we will do next is introduce some gauge conditions to simplify above equations.

2 Characteristic initial value problem based on N-P formulism

In last section, we give a brief introduction about N-P formulism. We have seen the equations are very complex and strongly coupled, but Eq.(5)-(37) are in a very general from, i.e. we don't use a special coordinate and special tetrad. In this section, we want to choose a special gauge(include coordinate and tetrad) and simplify these equations. For simplicity, we only consider vacuum case, i.e. $\Phi_{ij} = \Lambda = 0$.

2.1 Choice of the frame and coordinate

Suppose we have two light cones N_1 and N_2 in space-time (M, g), $N_1 \cap N_2 = S$ and S is a space-like 2-dim compact surface. On S, we choose the tetrad as following : l tangents to the generator of N_1 and n tangents to the generator of N_2, m and $\bar{m}$ tangents to S. Suppose (x^1, x^2) are coordinates on S, we generalize them to the whole N_2, i.e. along each generator of N_2, x^A are all constant. We use the affine parameter u of generator as the third coordinate on N_2 and choose $(m, \bar{m})$ on N_2 as the parallel transportation along generator of N_2(Note : this parallel transportation is in N_2). Based on this choice, we have coordinate system (u, x^A) and $\{n, m, \bar{m}\}$ on N_2. Using the normalization (1), we have a unique choice of l at each point of N_2. Because u is the affine parameter of generator of N_2, we know $\nu|_{N_2} = (\gamma + \bar{\gamma})|_{N_2} = 0$, the parallel transportation tells us $(\gamma - \bar{\gamma})|_{N_2} = 0$. Next question is how to leave the light cone N_2. We have l at each point of N_2. Considering the null geodesics generated by l, suppose the affine parameter is r, we use r as the forth coordinate in the space-time and (u, x^A) will be constant along each curve of l. About the tetrad, we parallelly transport the tetrad $\{l, n, m, \bar{m}\}$ along the curve of l. These choices tell us $\kappa = \varepsilon = \pi = 0$, $\tau = \bar{\alpha} + \beta$ and $\rho = \bar{\rho}$ because l is hyper-surface orthogonal.

For a summery, we write down our gauge choice in terms of N-P formulism [5, 6, 7]:

$$
\begin{aligned}
& l = \frac{\partial}{\partial r}, \quad n = \frac{\partial}{\partial u} + U\frac{\partial}{\partial r} + X^A\frac{\partial}{\partial x^A}, \quad m = \omega\frac{\partial}{\partial r} + \xi^A\frac{\partial}{\partial x^A}, \\
& On\ M, \quad \kappa = \varepsilon = \pi = \tau - \bar{\alpha} - \beta = \rho - \bar{\rho} = 0, \\
& On\ N_2, \quad U = X^A = \omega = \nu = \gamma = \mu - \bar{\mu} = 0. \qquad (38)
\end{aligned}
$$

2.2 Initial values

In ordinary Cauchy problem of Einstein equations, we know the initial data are $(h_{ij},\ k_{ij})$ on Σ, i.e. the first and second fundamental form of Cauchy surface Σ, which satisfies the constrain equations as

$$\begin{aligned} R_h - k\cdot k + (tr_h k)^2 &= 0, & (39)\\ D\cdot k - D(tr_h k) &= 0. & (40)\end{aligned}$$

Above equations are non-linear elliptic equations and is very difficult to find a solution. What we are interested is what is the real freedom of the initial data. In ordinary Cauchy surface, this question is very difficult to answer because of the constrain equations. In the characteristic initial value problem, things become more simple. Following theorem tells us the initial freedom of characteristic initial value is unconstrained[5, 6].

Theorem 1 *Let a "reduced initial data set" s_0 be a set of real and complex functions as following : Ψ_4 on N_2, Ψ_0 on N_1, $(\rho,\ \mu,\ \sigma,\ \lambda,\ \tau,\ \xi^A)$ on S. s_0 will uniquely determines a complete set of initial data $s = (e_i^\mu, \Gamma^\mu_{\alpha\beta}, \Psi_n)$ on $N_1 \cup N_2$.*

Proof : General speaking, the unknown functions of system (5)-(37) are

$$\begin{array}{ccccc} U & X^A & \omega & \xi^A & \\ \kappa & \varepsilon & \pi & \sigma & \rho \\ \tau & \gamma & \nu & \lambda & \mu \\ & \bar{\alpha}+\beta & \bar{\alpha}-\beta & & \\ \Psi_0 & \Psi_1 & \Psi_2 & \Psi_3 & \Psi_4 \end{array} \qquad (41)$$

Using the gauge conditions (38), we get the unknown functions are

$$\begin{array}{ccccc} U & X^A & \omega & \xi^A & \\ \sigma & \rho & \bar{\alpha}+\beta & \bar{\alpha}-\beta & \\ \gamma & \nu & \lambda & \mu & \\ \Psi_0 & \Psi_1 & \Psi_2 & \Psi_3 & \Psi_4 \end{array} \qquad (42)$$

Given the reduced initial data set s_0, we try to rediscover the complete initial data set s on $N_1 \cup N_2$.

1. On S, we have known $U = X^A = \omega = \nu = \gamma = 0$, ξ^A, ρ, σ, λ, μ, $\bar{\alpha}+\beta$, Ψ_0, Ψ_4; $\bar{\alpha}-\beta$ is the inner connection of two sphere, which can be got from ξ^A by standard calculation. From Eq.(15), (16) and (17), we can get the value of Ψ_1, Ψ_2 and Ψ_3. Now we have recover the complete set s on S.

2. On N_2, we know $U = X^A = \omega = \nu = \gamma = 0$ and the value of Ψ_4. We just get the complete set s o S. Along each generator of N_2, Eq.(14) and (18) reduces to a first order ordinary differential system

$$\begin{aligned} \frac{\partial\lambda}{\partial u} &= -2\mu\lambda - \Psi_4 & (43)\\ -\frac{\partial\mu}{\partial u} &= \mu^2 + |\lambda|^2 & (44)\end{aligned}$$

The basic theorem of ordinary differential equations insures the solution uniquely exists for given initial data. Similarly, we have the equation from commutator relation

$$-\frac{\partial\xi^A}{\partial u} = \bar{\lambda}\bar{\xi}^A + \mu\xi^A \qquad (45)$$

to fix the value of ξ^A on N_2. Then equations from structure equations and Bianchi identities

$$-\frac{\partial\beta}{\partial u} = \mu(\bar{\alpha}+\beta)+\beta\mu+\alpha\bar{\lambda} \tag{46}$$
$$\frac{\partial\alpha}{\partial u} = -(\bar{\alpha}+2\beta)\lambda-\mu\alpha-\Psi_3 \tag{47}$$
$$\frac{\partial\Psi_3}{\partial u} = \delta\Psi_4+(3\beta-\bar{\alpha})\Psi_4-4\mu\Psi_3 \tag{48}$$

fix the value of α, β and Ψ_3 on N_2,

$$-\frac{\partial\sigma}{\partial u} = -\delta\tau+\mu\sigma+\bar{\lambda}\rho+(\tau+\beta-\bar{\alpha})\tau \tag{49}$$
$$\frac{\partial\rho}{\partial u} = \bar{\delta}\tau-\rho\mu-\sigma\lambda+(\bar{\beta}-\alpha-\bar{\tau})\tau-\Psi_2 \tag{50}$$
$$\frac{\partial\Psi_2}{\partial u} = \delta\Psi_3+\sigma\Psi_4+2(\beta-\tau)\Psi_3-3\mu\Psi_2 \tag{51}$$

fix the value of ρ, σ and Ψ_2 on N_2,

$$\frac{\partial\Psi_1}{\partial u} = \delta\Psi_2-2\mu\Psi_1-3\tau\Psi_2+2\sigma\Psi_3 \tag{52}$$
$$\frac{\partial\Psi_0}{\partial u} = \delta\Psi_1-\mu\Psi_0-2(2\tau+\beta)\Psi_1+3\sigma\Psi_2 \tag{53}$$

fix the value of Ψ_1 and Ψ_0 on N_2. That is the complete set s on N_2.

3. On N_1, we can use the similar method to fix all components of s and finish the proof.

2.3 Reduced evolution equations

Using the gauge we have chosen, we can reduce the evolution equations (5)-(37) to following form,

$$DX^A = (\bar{\alpha}+\beta)\bar{\xi}^A+(\alpha+\bar{\beta})\xi^A, \tag{54}$$
$$DU = (\bar{\alpha}+\beta)\bar{\omega}+(\alpha+\bar{\beta})\omega-\gamma-\bar{\gamma}, \quad from\ Eq.(23), \tag{55}$$
$$D\xi^A = \rho\xi^A+\sigma\bar{\xi}^A, \tag{56}$$
$$D\omega = \rho\omega+\sigma\bar{\omega}-(\bar{\alpha}+\beta), \quad from\ Eq.(24), \tag{57}$$
$$D\rho = \rho^2+|\sigma|^2, \tag{58}$$
$$D\sigma = 2\rho\sigma+\Psi_0, \tag{59}$$
$$D\alpha = \alpha\rho+\beta\bar{\sigma}, \tag{60}$$
$$D\beta = \beta\rho+\alpha\sigma+\Psi_1, \tag{61}$$
$$D\gamma = \tau\alpha+\bar{\tau}\beta+\Psi_2, \tag{62}$$
$$D\lambda = \lambda\rho+\mu\bar{\sigma}, \tag{63}$$
$$D\mu = \mu\rho+\lambda\sigma+\Psi_2, \tag{64}$$
$$D\nu = \tau\lambda+\bar{\tau}\mu+\Psi_3, \tag{65}$$

$$\begin{aligned} & & from\ Eq.(5)-(22), \\ \nabla\cdot\Psi &=& B\cdot\Psi, \\ & & Bianchi\ idetities, \end{aligned} \tag{66}$$

where

$$\begin{aligned} \nabla &:= \begin{pmatrix} D' & -\delta & 0 & 0 & 0 \\ -\bar{\delta} & D+D' & -\delta & 0 & 0 \\ 0 & -\bar{\delta} & D+D' & -\delta & 0 \\ 0 & 0 & -\bar{\delta} & D+D' & -\delta \\ 0 & 0 & 0 & -\bar{\delta} & D \end{pmatrix}, \\ \Psi^T &:= (\Psi_0, \Psi_1, \Psi_2, \Psi_3, \Psi_4), \\ B &:= \begin{pmatrix} 4\gamma-\mu & -(4\tau+2\beta) & 3\sigma & 0 & 0 \\ \nu-4\alpha & 2\gamma-2\mu+4\rho & -3\tau & 2\sigma & 0 \\ -\lambda & 2\nu-2\alpha & 3\rho-3\mu & 2\beta-2\tau & \sigma \\ 0 & -2\lambda & 3\nu & 0 & 0 \\ 0 & 0 & -3\lambda & 2\alpha & \rho \end{pmatrix}. \end{aligned}$$

We want to ask the relation between the solution of reduced system and original system (5)-(37) under the gauge (38). A strict theorem tells us they are equivalent [5, 6].

Theorem 2 *If $\tilde{s} = (e_i^\mu,\ \Gamma^\mu_{\alpha\beta},\ \Psi_n)$ is a solution on M of the reduced system (54)-(66), which coincides on $N_1 \cup N_2$ with the initial data set s in theorem 1, then $\tilde{s}$ is a solution of system (5)-(37) under the gauge (38).*

Now we consider the reduced system (54)-(66). We find they can be written into a general form :

1. $C^\mu \partial_\mu s + F(s) = 0$;
2. C^μ are hermitian;
3. $C^\mu(e_{1\mu} + e_{2\mu})$ is positive definite;

so we know the reduced system is a quasi-linear sysmetric hyperbolic system and the solution uniquely exists.

Theorem 3 *Given analytic reduced initial data on $N_1 \cup N_2$, there exists a unique analytic solution of Einstein equations in the neighborhood of S, which implies the given data on $N_1 \cup N_2$.*

Remark:

1. Above result can be generalized to C^∞ solutions[8, 9].

2. The property that "C^μ are hermitian" and "$C^\mu(e_{1\mu} + e_{2\mu})$ is positive definite" will be needed to prove the formal series solution is convergent.

3 The application of characteristic initial value problem in quasi-local energy problem

It is well known that gravitational field has no local energy density, this property can be seen as the result of equivalent principle[10]. Because of the need of theoretical and numerical work, people try to define a quasi-local energy for gravitational field. About these work, there is a excellent review by L.B.Szabados [12].

In order to define the quasi-local energy, we consider a space-like finite region Σ in space-time, $S = \partial\Sigma$ is a compact 2-dim surface. The induced metric on Σ is h_{ab} and the induced metric on S is σ_{ab}. N^a is the time direction. $N^a = Nn^a + s^a$, where n^a is the normal vector of Σ, N is the lapse function and s^a is the shift vector.

Some criteria for the definition of quasi-local energy :

1. $E(S)$ only depends on data on S.
2. Positivity.
3. On an apparent horizon S_H, satisfies the Penrose inequality, i.e. $E(S_H) \geq \sqrt{\frac{Area(S_H)}{16\pi}}$.
4. $E(S) = 0$ in Minkowski space-time.
5. The limit at null and spatial infinity should give the Bondi mass and the ADM mass.

Some definition of quasi-local energy :

1. **Bartnik mass**

One of the most natural idea of quasi-localization of ADM mass is due to Bartnik(1989)[13]. Let $\mathcal{C}_k$ are initial surfaces and are all asymptotic flat, for each $\mathcal{C}_k$, exist a isomorphic embedding map $\phi_k : \Sigma \to \mathcal{C}_k$,

$$m_B(\Sigma) := \inf\{E_{ADM}(\mathcal{C}_k)\} \tag{67}$$

Obviously, m_B is finite, positive and monotonic. In Minkowski space-time, $m_B = 0$.

2. **Hawking mass**

Studying the perturbation of the dust-filled $k = -1$ FRW space-time(1968)[14], Hawking found he could define following quantity as the gravitational energy surrounded by the space-like topological 2-sphere S,

$$\begin{aligned} E_H(S) &:= \sqrt{\frac{A(S)}{16\pi}}(1 + \frac{1}{2\pi}\int_S \rho\rho' dS) \\ &= \sqrt{\frac{A(S)}{16\pi}}\frac{1}{2\pi}\int_S Re(\sigma\sigma' - \Psi_2 + \Phi_{11} + \Lambda)dS \end{aligned} \tag{68}$$

3. **Brown-York expression**

Based on the Hamilton-Jacobi method, Brown and York(1993) [15] define quasi-local energy for S as

$$E_{BY} := \int_S (k - k_0)dS \tag{69}$$

where S is embedded into 3-dim space-like surface $\mathcal{C}$, the normal direction of S in $\mathcal{C}$ is r^a, k is the extrinsic curvature of r^a in $\mathcal{C}$, k_0 is the reference term, it means we embed S into a back ground space-time isometrically.

4. **Kijowski's expression**

Under some suitable choice of boundary condition, Kijowski(1997) [17] define the quasi-local energy as

$$E_K := \frac{1}{16\pi}\int_S \frac{k_0^2 - (k^2 - l^2)}{k^0} dS \tag{70}$$

where l is the trace of the extrinsic curvature of time direction.

C.C.M.Liu and S.T Yau (2003) [18] give a modified Kijowski energy expression as

$$E_{KLY} := \frac{1}{16\pi}\int_S (k^0 - \sqrt{k^2 - l^2}) dS \tag{71}$$

About the reference term, they both embed the 2-sphere S into R^3 in Minkowski space-time. C.C.M.Liu and S.T.Yau have shown that E_{KLY} is positive under some convex conditions.

5. **Chen-Nester expression**

Instead of (h_{ab}, k_{ab}), Chen and Nester(1995)[19] use the orthonormal tetrad and connection as variable. Using the Hamilton-Jacobi method, they got the quasi-local energy and momentum as

$$p_\nu = \int_S (\Delta\omega^{ij}\ i_{N_\nu}\eta_{ij} - \overset{\circ}{\nabla}{}^i\ \overset{\circ}{N}_\nu{}^j\ i_{N_\nu}\Delta\eta_{ij})\ , \qquad \nu = 0, 1, 2, 3 \tag{72}$$

where $\overset{\circ}{\nabla}$ and $\overset{\circ}{N}{}^a$ are the connection and the time direction in the reference space-time, $\eta_{ab} = \epsilon_{abcd}\,\vartheta^c \wedge \vartheta^d$, $\Delta\eta_{ab} = \epsilon_{abcd}(\vartheta^c \wedge \vartheta^d - \overset{\circ}{\vartheta}{}^c \wedge \overset{\circ}{\vartheta}{}^d)$, $\{\vartheta^a\}$ and $\{\overset{\circ}{\vartheta}{}^a\}$ are the frames in physical and reference space-time.

3.1 The null infinity limit of Chen-Nester expression and it flux

Based on the basic criteria in last subsection, our resent work just want to consider null infinity limit of Chen-Nester's energy expression under several known embedding ways.

3.1.1 Asymptotic behavior of space-time near null infinity

In order to consider the null infinity limit of the energy-momentum and their flux, let's firstly consider the asymptotic behavior of the space-time near the null infinity. In this work we are interested on the cases in which the space-time is asymptotically flat. There exist many intensive papers on this subject in the past decades. We will follow the method initiated by Newman, Penrose, and Tod [4, 22]. Many useful results have been discussed in [22]-[26].

According to the Penrose's conformal compact method, i.e. the Penrose diagram, we assume that $(\tilde{M},\ \tilde{g}_{\mu\nu},\ \Omega)$ is the conformal manifold of a physical space-time $(M,\ g_{\mu\nu})$ via a conformal transformation $\tilde{g}_{\mu\nu} = \Omega^2\, g_{\mu\nu}$. Suppose S is a section of future null infinity $\mathcal{I}^+$, we choose the *Bondi coordinates* near $\mathcal{I}^+$ in the following way: $(\theta,\ \phi)$ is a spherical coordinate on S and u is the affine parameter generating $\mathcal{I}^+$ such that $u = 0$ on S. If k^μ is a null vector on $\mathcal{I}^+$ which is not tangential, namely $k^\mu \notin T(\mathcal{I}^+)$, the null geodesics generated by k^μ will take the coordinates $(u,\ \theta,\ \phi)$ into the physical space-time $(M,\ g_{\mu\nu})$. Moreover, on each null geodesic $\psi(r) = \{u = \text{const}, \theta = \text{const}, \phi = \text{const}\}$, its affine parameter r, in the sense of the physical metric $g_{\mu\nu}$, can serve as the forth coordinate (in the neighborhood of $\mathcal{I}^+$, we assume the null geodesics are future complete). Finally we have constructed the Bondi coordinates (u, r, θ, ϕ) in the physical space-time.

Beside the Bondi coordinates construction we impose, as have introduced in [25, 26], the following gauges

$$\rho - \bar{\rho} = \mu - \bar{\mu} = \kappa = \varepsilon = \tau - \bar{\alpha} - \beta = \bar{\pi} - \bar{\alpha} - \beta = 0. \tag{73}$$

By introducing a complex coordinate on the two sphere by

$$\zeta := \cot\frac{\theta}{2}\,\mathrm{e}^{i\phi}, \tag{74}$$

the null tetrad can be chosen as

$$\begin{aligned} l^a &= \frac{\partial}{\partial r}, && (75)\\ n^a &= \frac{\partial}{\partial u} + U\frac{\partial}{\partial r} + X\frac{\partial}{\partial \zeta} + \bar{X}\frac{\partial}{\partial \bar{\zeta}}, && (76)\\ m^a &= \xi\frac{\partial}{\partial \zeta} + \bar{\eta}\frac{\partial}{\partial \bar{\zeta}}, && (77) \end{aligned}$$

where U (real) and X, ξ, η (complex) are undetermined functions.

For simplicity, in this paper we will only focus on the vacuum case. Due to the asymptotic flatness, the behavior of the Weyl curvature satisfies the "Peeling off theorem" by Penrose [2, 3], i.e.

$$\Psi_n \sim O(r^{n-5}), \qquad n = 0, 1, 2, 3, 4. \tag{78}$$

Near the null infinity, we expand all quantities in the Taylor series of r. Using the Newman-Penrose equations, the asymptotic behavior of the spin-coefficients are

$$\begin{aligned} \rho &= -\frac{1}{r} - \frac{|\sigma^0|^2}{r^3} + O(r^{-5}), && (79)\\ \sigma &= \frac{\sigma^0}{r^2} + \frac{|\sigma^0|^2\sigma^0 - \frac{1}{2}\Psi_0^0}{r^4} + O(r^{-5}), && (80)\\ \alpha &= \frac{\alpha^0}{r} + \frac{\bar{\sigma}^0\bar{\alpha}^0 + \eth_0\bar{\sigma}^0}{r^2} + O(r^{-3}), && (81)\\ \beta &= -\frac{\bar{\alpha}^0}{r} - \frac{\alpha^0\sigma^0}{r^2} + O(r^{-3}), && (82)\\ \tau &= \frac{\bar{\eth}_0\sigma^0}{r^2} + O(r^{-3}) = \bar{\pi}, && (83)\\ \gamma &= \frac{\gamma_1}{r^2} + O(r^{-3}) = \frac{-\alpha^0\bar{\eth}_0\sigma^0 + \bar{\alpha}^0\eth_0\bar{\sigma}^0 - \frac{1}{2}\Psi_2^0}{r^2} + O(r^{-3}), && (84)\\ \mu &= -\frac{1}{2r} - \frac{\Psi_2^0 + \sigma^0\dot{\bar{\sigma}}^0 + \eth_0^2\bar{\sigma}^0}{r^2} + O(r^{-3}), && (85)\\ \lambda &= \frac{\dot{\bar{\sigma}}^0}{r} + \frac{\frac{1}{2}\bar{\sigma}^0 - \bar{\eth}_0\eth_0\bar{\sigma}^0}{r^2} + O(r^{-3}), && (86) \end{aligned}$$

$$
\begin{aligned}
\nu &= \frac{\nu^1}{r^2} + O(r^{-3}), && (87)\\
U &= -\frac{1}{2} + \frac{\gamma_1 + \bar{\gamma}_1}{r} + O(r^{-2}), && (88)\\
X &= -\frac{\eth_0 \bar{\sigma}^0 \xi^0}{r^2} + O(r^{-3}), && (89)\\
\xi &= \frac{\xi^0}{r} + \frac{|\sigma^0|^2 \xi^0}{r^3} + O(r^{-4}), && (90)\\
\eta &= -\frac{\bar{\sigma}^0 \xi^0}{r^2} + O(r^{-4}). && (91)
\end{aligned}
$$

where $\sigma^0(u,\,\theta,\,\phi)$, $\Psi_2^0(u,\,\theta,\,\phi)$ are two unspecified functions and $\alpha^0 := \frac{-1}{2\sqrt{2}}\zeta$. The operator $\eth_0$ is the spin-weight operator[3, 23], $\eth_0 f := \left(\frac{P}{\sqrt{2}}\frac{\partial}{\partial \zeta} + 2s\bar{\alpha}^0\right) f$ for any quantity f which has spin-weight s and $P := 1 + \zeta\bar{\zeta} = \sin^{-2}\frac{\theta}{2}$. $\xi^0 = \frac{P}{\sqrt{2}}$. Consider the induced metric ${}^{(2)}ds^2$ on the sphere $S = \{u = \text{const},\ t = \text{const}\}$, remember $\frac{\partial}{\partial\zeta} = -\sin^2\frac{\theta}{2}\,\mathrm{e}^{-i\phi}(\frac{\partial}{\partial\theta} + \frac{i}{\sin\theta}\frac{\partial}{\partial\phi})$, we get

$$
{}^{(2)}ds^2 = r^2(d\theta^2 + \sin^2\theta\, d\phi^2) + O(r), \tag{92}
$$

and the induced 2-dimensional volume element on S^2 is

$$
\begin{aligned}
{}^2\epsilon_{cd} &= \epsilon_{abcd}\left(\frac{\partial}{\partial u}\right)^a \left(\frac{\partial}{\partial r}\right)^b \\
&= i\ \mathbf{m} \wedge \bar{\mathbf{m}} + O(r) \\
&= r^2\left(1 - \frac{|\sigma^0|^2}{r^2}\right) d\Omega_0^2 + O(r^{-2}). && (93)
\end{aligned}
$$

3.1.2 Several embedding method

Usually, we will choose the reference space as the Minkowski space-time. In our work, we will use the Eddington-Finkelstein coordinates $(u,\,r,\,\theta,\,\phi)$ of Minkowski space, which is related to the standard coordinates by

$$
u = t - r, \tag{94}
$$

the first fundamental form of Minkowski space-time is

$$
ds^2 = -du^2 - 2dudr + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{95}
$$

It is easy to verify that the coordinates we used is just the Bondi coordinates.

However, the issue of embedding the Minkowski space-time into physical space-time is not fully transparent yet. In this section, we consider several types of embedding.

3.2 The holonomic embedding

In this method, we just identify the Bondi-coordinates with the Minkowski coordinates, i.e. the embedded surface S_0 is just the standard coordinate sphere. The embedded null frame are

$$\mathring{l}^a = \frac{\partial}{\partial r}, \tag{96}$$

$$\mathring{n}^a = \frac{\partial}{\partial u} - \frac{1}{2}\frac{\partial}{\partial r}, \tag{97}$$

$$\mathring{m}^a = \frac{1}{\sqrt{2}r}\, e^{-i\phi}\left(\frac{\partial}{\partial\theta} + \frac{i}{\sin\theta}\frac{\partial}{\partial\phi}\right) = \frac{P}{\sqrt{2}r}\frac{\partial}{\partial\zeta}. \tag{98}$$

Under this choice, we know the N-P coefficients in reference space are

$$\mathring{\rho} = -\frac{1}{r}\ ,\ \mathring{\mu} = -\frac{1}{2r}\ ,\ \mathring{\alpha} = -\mathring{\bar{\beta}} = -\frac{\zeta}{2\sqrt{2}r}\ ,\ \textit{others are all zero} \tag{99}$$

The embedding of the time direction $\mathring{N}^{\ a}$ is $\mathring{N}^{\ a} = \frac{1}{2}\mathring{l}^a + \mathring{n}^a$, we know such $\mathring{N}^{\ a}$ is the Killing vector of Minkowski space-time.

3.3 N.O'Murchadha, L.B.Szabados and K.P.Tod's embedding

In ref.[30], N.O'Murchadha, L.B.Szabados and K.P.Tod introduce another kind of embedding method. Let's consider the space-like region Σ_0 in physical space-time, $\partial\Sigma_0 = S$, we suppose S is a topological two sphere and the isothermal coordinates globally exist on S. Under this coordinates, the induced metric ${}^{(2)}ds^2$ on S is

$${}^{(2)}ds^2 = \omega^2(d\theta^2 + \sin\theta^2 d\phi^2) \tag{100}$$

Where ω is a positive function on S. The embedding method is

$$u = const.,\ \ r = \omega(\theta,\phi),\ \ \theta = \theta \ \textit{and}\ \phi = \phi \tag{101}$$

Based on this embedding, we define a coordinate transformation in Minkowski space-time,

$$\begin{aligned} U &= u \\ R &= r - \omega \\ \theta &= \theta \\ \phi &= \phi \end{aligned} \tag{102}$$

The Minkowski metric becomes

$$(\mathring{g}_{\mu\nu}) = \begin{pmatrix} -1 & -1 & -\frac{\partial\omega}{\partial\theta} & -\frac{\partial\omega}{\partial\phi} \\ -1 & 0 & 0 & 0 \\ -\frac{\partial\omega}{\partial\theta} & 0 & (R+\omega)^2 & 0 \\ -\frac{\partial\omega}{\partial\phi} & 0 & 0 & (R+\omega)^2\sin^2\theta \end{pmatrix} \tag{103}$$

We choose the N-P tetrad as

$$\begin{aligned} \mathring{l}^{\ a} &= \frac{\partial}{\partial R}, \\ \mathring{n}^{\ a} &= \frac{\partial}{\partial U} - \frac{1}{2}\left(1 + \frac{|\delta_0\omega|^2}{(R+\omega)^2}\right)\frac{\partial}{\partial R} + \frac{1}{(R+\omega)^2}\frac{\partial\omega}{\partial\theta}\frac{\partial}{\partial\theta} + \frac{1}{(R+\omega)^2\sin^2\theta}\frac{\partial\omega}{\partial\phi}\frac{\partial}{\partial\phi}, \\ \mathring{m}^{\ a} &= \frac{e^{-i\phi}}{\sqrt{2}(R+\omega)}\left(\frac{\partial}{\partial\theta} + \frac{i}{\sin\theta}\frac{\partial}{\partial\phi}\right), \end{aligned} \tag{104}$$

where $\delta_0 = \frac{\partial}{\partial\theta} + \frac{i}{\sin\theta}\frac{\partial}{\partial\phi}$. From above metric, direct calculation gives the N-P coefficients are

$$\begin{aligned}
\mathring{\varepsilon} &= 0, && (105)\\
\mathring{\kappa} &= 0, && (106)\\
\mathring{\rho} &= -\frac{1}{R+\omega}, && (107)\\
\mathring{\sigma} &= 0, && (108)\\
\mathring{\mu} &= -\frac{1}{2}\left[\frac{1}{R+\omega} + \frac{|\delta_0\omega|^2}{(R+\omega)^3} - \frac{\tilde{\Delta}\omega}{(R+\omega)^2}\right], && (109)\\
\mathring{\lambda} &= \frac{e^{2i\phi}}{2(R+\omega)^2}\left[\bar{\delta}_0^2\omega - \cot\theta\bar{\delta}_0\omega - \frac{2(\bar{\delta}_0\omega)^2}{R+\omega}\right], && (110)\\
\mathring{\tau} &= \mathring{\pi} = \mathring{\alpha} + \bar{\mathring{\beta}} = -\frac{e^{i\phi}\bar{\delta}_0\omega}{\sqrt{2}(R+\omega)^2} && (111)\\
\mathring{\alpha} &= -\frac{e^{i\phi}}{2\sqrt{2}}\left[\frac{\cot\frac{\theta}{2}}{R+\omega} + \frac{2\bar{\delta}_0\omega}{(R+\omega)^2}\right], && (112)\\
\mathring{\beta} &= \frac{e^{i\phi}}{2\sqrt{2}}\frac{\cot\frac{\theta}{2}}{R+\omega}, && (113)\\
\mathring{\nu} &= -\frac{e^{i\phi}}{\sqrt{2}(R+\omega)^3}\left[\frac{|\delta_0\omega|^2\bar{\delta}_0\omega}{R+\omega} - \delta_0\omega\left(\bar{\delta}_0\partial_\theta\omega + \frac{i\cot\theta\partial_\phi\omega}{\sin\theta}\right) + \frac{i\partial_\phi\omega}{\sin\theta}\tilde{\Delta}\omega\right], && (114)\\
\mathring{\gamma} &= -\frac{1}{2}\left[\frac{|\delta_0\omega|^2}{(R+\omega)^3} + \frac{i\cot\frac{\theta}{2}\partial_\phi\omega}{(R+\omega)^2\sin\theta}\right], && (115)
\end{aligned}$$

where $\tilde{\Delta}$ is the laplace operator on the coordinate two sphere. The image of S is the two sphere in Minkowski space-time such that $U = const.$ and $R = 0$.

3.4 Brown, Lau and York's embedding

In above two subsections, the two method we used all embeds the S into a standard light cone in Minkowski space-time, i.e. the light cone N is the light cone from one point. In ref.[31], J.D.Brown, S.R.Lau and J.W.York give another way to consider the embedding near null infinity. This method consider a more general light cone.

Suppose we choose a Bondi coordinate system (u, R, θ, ϕ) in Minkowski space-time, the asymptotic shear is $\mathring{\sigma}^0$. We can also do the formal Taylor extension near the null infinity of Minkowski space-time as we have done before. The only difference is we have the requirement $\Psi_n = 0,\ n = 0, 1, 2, 3, 4$. Using N-P equations, we have

$$\begin{aligned}
\Psi_3^0 &= -\eth_0\dot{\mathring{\sigma}}^0\\
\Psi_4^0 &= -\ddot{\mathring{\sigma}}^0 && (116)
\end{aligned}$$

Because the spin-weight of $\dot{\mathring{\sigma}}^0$ is non-zero, above results tell us that $\dot{\mathring{\sigma}}^0 = 0$ [?]. Submit this results into Eq.(5)-(87), the N-P quantities in Minkowski space-time should be

$$\mathring{\kappa} = 0 \qquad (117)$$

$$\overset{\circ}{\varepsilon} = 0 \tag{118}$$

$$\overset{\circ}{\rho} = -\frac{1}{R} - \frac{|\overset{\circ}{\sigma}{}^0|^2}{R^3} + O(R^{-5}) \tag{119}$$

$$\overset{\circ}{\sigma} = \frac{\overset{\circ}{\sigma}{}^0}{R^2} + \frac{|\overset{\circ}{\sigma}{}^0|^2\overset{\circ}{\sigma}{}^0}{R^4} + O(R^{-5}) \tag{120}$$

$$\overset{\circ}{\alpha} = \frac{\alpha^0}{R} + \frac{\bar{\overset{\circ}{\sigma}}{}^0\bar{\alpha}^0 + \eth_0\bar{\overset{\circ}{\sigma}}{}^0}{R^2} + O(R^{-3}) \tag{121}$$

$$\overset{\circ}{\beta} = -\frac{\bar{\alpha}^0}{R} - \frac{\alpha^0\overset{\circ}{\sigma}{}^0}{R^2} + O(R^{-3}) \tag{122}$$

$$\overset{\circ}{\tau} = \bar{\overset{\circ}{\pi}} = \frac{\bar{\eth}_0\overset{\circ}{\sigma}{}^0}{R^2} + O(R^{-3}) \tag{123}$$

$$\overset{\circ}{\gamma} = \frac{\overset{\circ}{\gamma}_1}{R^2} + O(R^{-3}) = \frac{-\alpha^0\bar{\eth}_0\overset{\circ}{\sigma}{}^0 + \bar{\alpha}^0\eth_0\bar{\overset{\circ}{\sigma}}{}^0}{R^2} + O(R^{-3}) \tag{124}$$

$$\overset{\circ}{\mu} = -\frac{1}{2R} - \frac{\eth_0^2\bar{\overset{\circ}{\sigma}}{}^0}{R^2} + O(R^{-3}) \tag{125}$$

$$\overset{\circ}{\lambda} = \frac{\frac{1}{2}\bar{\overset{\circ}{\sigma}}{}^0 - \bar{\eth}_0\eth_0\bar{\overset{\circ}{\sigma}}{}^0}{R^2} + O(R^{-3}) \tag{126}$$

$$\overset{\circ}{\nu} = O(R^{-3}) \tag{127}$$

The tetrad part should be

$$\overset{\circ}{U} = -\frac{1}{2} - \frac{\overset{\circ}{\gamma}_1 + \bar{\overset{\circ}{\gamma}}_1}{R} + O(R^{-2}) \tag{128}$$

$$\overset{\circ}{X} = -\frac{\bar{\eth}_0\overset{\circ}{\sigma}{}^0\xi^0}{R^2} + O(R^{-3}) \tag{129}$$

$$\overset{\circ}{\xi} = \frac{\xi^0}{R} + \frac{|\overset{\circ}{\sigma}{}^0|^2\xi^0}{R^3} + O(R^{-4}) \tag{130}$$

$$\overset{\circ}{\eta} = -\frac{\bar{\overset{\circ}{\sigma}}{}^0\xi^0}{R^2} + O(R^{-4}) \tag{131}$$

This embedding keeps the inner geometry of S unchanged, so we have

$$-\mu\rho + \lambda\sigma + \Psi_2 + \bar{\Psi}_2 + \bar{\lambda}\bar{\sigma} - \bar{\mu}\bar{\rho} = \frac{1}{2}\mathcal{R} = \frac{1}{2}\overset{\circ}{\mathcal{R}} = -\overset{\circ}{\mu}\overset{\circ}{\rho} - \bar{\overset{\circ}{\mu}}\bar{\overset{\circ}{\rho}} + \overset{\circ}{\lambda}\overset{\circ}{\sigma} + \bar{\overset{\circ}{\lambda}}\,\bar{\overset{\circ}{\sigma}} \tag{132}$$

Using the Taylor extension in Eq.(5)-(87) and Eq.(117)-(127), we get the relation between the parameter r and R is

$$\begin{aligned} R &= r + k + O(r^{-1}), \\ k &= \eth_0^2\bar{\overset{\circ}{\sigma}}{}^0 + \bar{\eth}_0^2\overset{\circ}{\sigma}{}^0 - \eth_0^2\bar{\sigma}^0 - \bar{\eth}_0^2\sigma^0 \end{aligned} \tag{133}$$

If we choose $\sigma^0|_S = \overset{\circ}{\sigma}{}^0|_{S_0}$, where S is the section on $\mathcal{I}^+$ and S_0 is its image under the embedding. We have

$$R = r + O(r^{-1}) = r[1 + O(r^{-2})] \tag{134}$$

3.4.1 Energy-momentum at null infinity

We have know the covariant quasi-local energy and momentum are

$$p_\nu = \int_S (\Delta\omega^{ij} i_{N_\nu}\eta_{ij} - \overset{\circ}{\nabla}{}^{\alpha} \overset{\circ}{N}{}^{\beta} i_{N_\nu}\eta_{\alpha\beta}) \tag{135}$$

In asymptotic flat case, N_ν should be the translation part of asymptotic Killing vectors. We have know the translation part of BMS group is well-defined, it should be

$$\begin{aligned}
N_\nu &= N_\nu^0 + O(r^{-1}), \quad \nu = 0,1,2,3 \\
N_0^0 &= \frac{\partial}{\partial u} = f_0 \frac{\partial}{\partial u} = f_0(n + \frac{1}{2}l + O(r^{-1})), \\
N_1^0 &= -\sin\theta\cos\phi(\frac{\partial}{\partial u} - \frac{\partial}{\partial r}) = f_1(n - \frac{1}{2}l + O(r^{-1})), \\
N_2^0 &= -\sin\theta\sin\phi(\frac{\partial}{\partial u} - \frac{\partial}{\partial r}) = f_2(n - \frac{1}{2}l + O(r^{-1})), \\
N_3^0 &= -\cos\theta(\frac{\partial}{\partial u} - \frac{\partial}{\partial r}) = f_3(n - \frac{1}{2}l + O(r^{-1}))
\end{aligned} \tag{136}$$

Submit above results into Eq.(135), we get

1. Under the holonomic embedding, the result is

$$\lim_{\mathcal{I}^+} p_\nu(S) = -\frac{1}{8\pi}\int_S Re(\Psi_2^0 + \sigma^0\dot{\bar{\sigma}}^0 + \eth_0^2\bar{\sigma}^0) f_\nu d\Omega_0 \tag{137}$$

2. Under the MST embedding, the result is

$$\lim_{\mathcal{I}^+} p_\nu(S) = -\frac{1}{8\pi}\int_S Re(\Psi_2^0 + \sigma^0\dot{\bar{\sigma}}^0 + \eth_0^2\bar{\sigma}^0) f_\nu d\Omega_0 \tag{138}$$

3. Under the BLY embedding, the result is

$$\lim_{\mathcal{I}^+} p_\nu(S) = -\frac{1}{8\pi}\int_S Re(\Psi_2^0 + \sigma^0\dot{\bar{\sigma}}^0) f_\nu d\Omega_0 \tag{139}$$

From above results, we find Eq.(135) gives the standard Bondi energy-momentum under the BLY embedding. In holonomic embedding and MST embedding, what we get is the Bondi energy-momentum adds a super-translation part. This additional part is caused by the embedding. In general case, the section S is not in a standard light cone, here "standard" means the light cone is generated by the null geodesics which escape from a single point. This may tell us we need more requirements than just keeping the inner geometry of S unchanged.

3.4.2 The energy flux near null infinity

We have known the Chen-Nester expression of the quasi-local energy-momentum on a two sphere S is

$$p_\nu = \frac{1}{16\pi}\int_S \left(\Delta\omega^{ab} \wedge i_{N_\nu}\eta_{ab} + \overset{\circ}{\nabla}{}^a \overset{\circ}{N}_\nu{}^b \Delta\eta_{ab}\right). \tag{140}$$

Suppose Σ_0 is the space-like region that we want to consider, $\partial\Sigma_0 = S$, N^a is the time direction of the "3+1" decomposition, $\Sigma_{\Delta t}$ is the time evolution of Σ_0 , it is easy to see the energy-momentum flowing into this region during the time Δt should be $p_\nu(\Sigma_{\Delta t}) - p_\nu(\Sigma_0)$, so there is a natural definition of the energy-momentum flux as

$$
\begin{aligned}
F_\nu &:= \lim_{\Delta t\to 0} \frac{p_\nu(\Sigma_{\Delta t}) - p_\nu(\Sigma_0)}{\Delta t} \\
&= \frac{1}{16\pi}\int_S \pounds_{N_0}(\Delta\omega^{ab}\wedge i_{N_\nu}\eta_{ab} + \mathring{\nabla}^a \mathring{N}_\nu{}^b \Delta\eta_{ab}).
\end{aligned} \tag{141}
$$

This expression also includes two parts which will be considered separately: the pure physical part and the other part including reference, i.e.

$$F_\nu = F_\nu^{\mathrm{phy}} + F_\nu^{\mathrm{ref}}, \tag{142}$$

where

$$
\begin{aligned}
F_\nu^{\mathrm{phy}} &:= \frac{1}{16\pi}\int_S \pounds_{N_0}(\omega^{ab}\wedge i_{N_\nu}\eta_{ab}), \\
F_\nu^{\mathrm{ref}} &:= -\frac{1}{16\pi}\int_S \pounds_{N_0}(\mathring{\omega}^{ab}\wedge i_{N_\nu}\eta_{ab}) + \frac{1}{16\pi}\int_S \pounds_{N_0}(\mathring{\nabla}^a \mathring{N}_\nu{}^b \Delta\eta_{ab}).
\end{aligned} \tag{143}
$$

We consider the null infinity limit of above energy flux. Some detailed calculation shows above three different embedding ways all get the standard Bondi energy-momentum loss formula [2, 3, 22, 14], i.e.

$$\lim_{\mathcal{I}^+} F_\nu = -\frac{1}{4\pi}\int_S |\dot{\sigma}^0|^2 f_\nu d\Omega_0. \tag{144}$$

References

[1] D. Kramer, H. Stephani, E. Herlt and M. MacCallum, *Exact Solutions of Einstein's Field Equations*, Cambridge University Press, 1980

[2] R. Penrose and W. Rindle, *Spinors and Space-time* Vol.1 and Vol.2, Cambridge University Press, 1984.

[3] J. Stewart, *Advanced General Relativity*, Cambridge University Press, 1990.

[4] E. T. Newman and R. Penrose, "An approach to gravitational radiation by a method of spin coefficients", J. Math. Phys. **3** (1962) 566.

[5] H. Fridrich, "On the regular and the asymptotic characteristic initial value problem for Einstein's vacuum field equations", Proc. R. Soc. Lond. **A 375** (1981) 169.

[6] H. Fridrich, "The asymptotic characteristic initial value problem for Einstein's vacuum field equations as an initial value problem for a first order quasi-linear symmetric hyperbolic system", Proc. R. Soc. Lond. **A 378** (1981) 401.

[7] H. Fridrich, "On the existence of analytic null asymptotically flat solutions of Einstein's vacuum field equations", Proc. R. Soc. Lond. **A 381** (1982) 361.

[8] J. Kánnár, "On the existence of C^∞ solutions to the asymptotic characteristic initial value problem in general relativity", Proc. R. Soc. Lond. **A 452** (1996) 945.

[9] A. D. Rendall, "Reduction of the characteristic initial value problem to Cauchy problem and its applications to Einstein equations", Proc. R. Soc. Lond. **A 427** (1990) 221.

[10] C.W.Misner, K.S.Thorne and J.A.Wheeler, *Gravitation*, Freeman, San Francisco, 1973.

[11] T.Regge and C.Teitelboim, Ann. Phys. **88** (1974) 286.

[12] L.B.Szabados, Livingreviews/lrr-2004-4.

[13] R.Bartnik, Phys.Rev.Lett. **62** (1989) 2346.

[14] S.W.Hawking, J.Math.Phys. **9** (1968) 598.

[15] J.D.Brown and J.M.York, Phys.Rev. **D 47** (1993) 1407.

[16] Y.Shi and L.F.Tam, J.Differ.Geom. **62** (2002) 79.

[17] J.Kijowski, G.R.G. **29** (1997) 307.

[18] C.C.M.Liu and S.T.Yau, Phys.Rev.Lett. **90** (2003) 231102.

[19] C.M.Chen, J.M.Nester and R.S.Tung, Phys.Lett. A **203** (1995) 5.

[20] C.C.Chang, J.M.Nester and C.M.Chen, Phys. Rev. Lett. **83**(1999)1897 [arXiv:gr-qc/9809040].

[21] R. Penrose, Proc. Roy. Soc. Lond. A **381**, 53-63 (1982).

[22] E.T.Newman and K.P.Tod, Asymptotically Flat Space-time, *General Relativity and Gravitation*, Vol.2, ed. by A.Held (New York: Plenum), 1980.

[23] R.Geroch, A.Held and R.Penrose, J.Math.Phys. **14** (1973) 874.

[24] M.Ludvigsen and J.A.G.Vickers, J.Phys.A : Math,Gen **16** (1983) 1155.

[25] R.M.Kelly, K.P.Tod and N.M.J.Woodhouse, Class. Quant. Grav. **3** (1986) 1151.

[26] A.Dougan, Class.Quant.Grav. **9** (1992) 2461.

[27] C.M.Chen and J.M.Nester, Class.Quant.Grav. **16** (1999) 1279.

[28] C.M.Chen and J.M.Nester, Grav.Cosmol. **6** (2000) 257.

[29] J.D.Brown, S.R.Lau, and J.W.York, Phys. Rev. **D 59** (1999) 064028.

[30] N.O'Murchadha, L.B.Szabados and K.P.Tod, Phys. Rev. Lett. **90**(2004) 259001.

[31] J.D.Brown, S.R.Lau and J.W.York, Phys. Rev. **D 55**(1997) 1977.