Tackling Confinement with Duality

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Outline

- **Introduction to Confinement**
- **Aspects of Supersymmetry**
- **Electric-Magnetic Duality**
- **Seiberg-Witten Theory**
- **Mechanism for Confinement in $N=1$ Supersymmetric Yang-Mills Theory**
- **Summary**

1 Introduction to Confinement in QCD

Three Prominent Features of QCD

- Asymptotic freedom (2004 Nobel Prize)
- Color confinement
- Chiral symmetry breaking

Color confinement:

- no asymptotic state for color-charged particle in QCD;
- observed hadrons are color singlets;
- microscopic variable $\neq$ macroscopic observable

Explicit dynamical Dynamical mechanisms for confinement is **not clear**!

Difficulty:

- **Qualitative reason:**
  - Strong coupling at low-energy (below Λ_{QCD});
  - Perturbative calculation does not work;
  - no way for analytical solution
- **Quantitative reason:** Complicated vacuum structure
  - Degenerate classical topological vacua configuration

- Tunneling effect by instanton $\Longrightarrow$ 't Hooft effective action
- Axial $U_A(1)$ (perturbative and non-perturbative) anomalous effective action
- Strong coupling $\Longrightarrow$ Quark condensation; gluon condensation and various exotic states
- ...

$\Longrightarrow$ Low-energy QCD has very complicated phase structure; partially shown by lattice stimulation

Phenomenological interpretation for confinement mechanism (before 1994):

Dual (type II) superconductivity mechanism of QCD vacuum (**Nambu, 1974; Mandelstam, 1976; ' t Hooft, 1981**)

- **Meissner effect in Type II Superconductivity:** from field theory viewpoint

  - In a superconductor, electron condensation $\Longrightarrow$ spontaneous breaking of $U(1)$ electromagnetic gauge symmetry $\longleftrightarrow$ massive magnetic field $\longleftrightarrow$ Meissner effect
  - **Meissner effect** (diamagnetism):
    * Outside superconductor, exclusion of magnetic field from entry into superconductor;

  * Inside superconductor, magnetic flux are pressed into very narrow tube and inside tube it is normal state $\longrightarrow$ energy carried by tube per unit length is constant

- Theoretical model: **Landau-Ginzburg theory** (**Ginzburg and Landau, 1950**)

$$
\begin{aligned}
S &= \int d^3x \left[-\frac{1}{4m} \left| \left(\nabla \psi_i - 2ie\mathbf{A} \right) \psi \right|^2 \right. \\
&\quad \left. -\alpha(T) \left| \psi(\mathbf{x}) \right|^2 - \frac{1}{2} \beta \left| \psi(\mathbf{x}) \right|^4 \right] \qquad (1)
\end{aligned}
$$

$\psi(\mathbf{x})$ —- wave function for superconducting electron

$\mathbf{A}$ —- external magnetic potential

  * Using the Landau theory of phase transition to treat the transition from normal state to superconducting state
  * Specifically magnetic vertex structure (mixed or intermediate state) were predicted by Abrikosov within this model
  * Macroscopic description to BCS theory and description only on the superconducting electron
  * **a masterstroke of modern physics**
  * Gauge invariance

$$
\begin{aligned}
\mathbf{A}(\mathbf{x}) &\rightarrow \mathbf{A}(\mathbf{x}) + \nabla \varphi(\mathbf{x}), \\
\psi(\mathbf{x}) &\rightarrow \exp\left[i \frac{2\pi}{\Phi_0} \varphi(\mathbf{x}) \right] \psi(\mathbf{x}), \\
\Phi_0^2 &= \frac{\alpha(T)}{\beta} \qquad (2)
\end{aligned}
$$

* When $|\psi| \neq 0$, $U(1)$ symmetry breaking, Cooper electron pair condensation $\Longrightarrow$

  Phase transition from normal state to superconducting state

* **Type-II superconductor:**

$$\xi(T) < \sqrt{2}\lambda(T), \quad \xi^2 = \frac{1}{4m\alpha(T)}, \quad \lambda^2 = \frac{m\beta}{8\pi e^2\alpha(T)} \qquad (3)$$

  ξ — **Coherence length** = distance over which $\psi(r)$ decrease to zero but without undue energy increase in the interface between superconducting and normal domain

  λ — **penetration depth** into superconductor of magnetic field in Meissner effect

* **Magnetic flux tube= Abrikosov vertex solution to linearized L-G equation**

  Linearized Landau-Ginzburg equation

$$\frac{1}{4m}(\nabla - 2ie\mathbf{A})\psi_{\mathrm{L}}(\mathbf{x}) = \alpha\psi_{\mathrm{L}}(\mathbf{x}),$$
$$\nabla \times \mathbf{A} = (0, 0, H_{\mathrm{h}}) \qquad (4)$$

  General solution (Abrikosov vertex state near H_{h}:

$$\begin{aligned}
\psi_{\mathrm{L}}(\mathbf{x}) &= \sum_n C_n \exp(inqy) \exp\left[-\frac{1}{2}\left(\frac{x - x_n}{\xi(T)}\right)\right], \\
x_n &= \frac{nq}{2eH_{\mathrm{h}}}, \quad \Delta x \Delta y H_{\mathrm{h}} = \Phi_0, \\
q &= \frac{2\pi}{\Delta y}, \quad \nu \in Z
\end{aligned} \qquad (5)$$

- **Dual superconductivity:**
  - electric field $\longrightarrow$ magnetic field
  - electron condensation $\longrightarrow$ magnetic monopole condensation $\Longrightarrow$
    * Spontaneous breaking of electric $U(1)$ symmetry $\longrightarrow$ Spontaneous breaking of magnetic $U(1)$ symmetry

- **Dual superconductivity of QCD vacuum**
  - QCD vacuum medium expel chromoelectric field
  - Chromoelectric flux form thin tube
    $\Longrightarrow$ Condensation of chromomagnetic monopole = Dual Meissner effect

$\Longrightarrow$ color charge confinement

- **History on the development of this confinement mechanism**
  - **Nielsen-Olesen vortex solution** in Abelian Higgs theory = **dual closed string** (**Nielsen** & **Olesen, 1973**)
    * Abelian Higgs theory $=U(1)$ gauge field + complex scalar field with symmetry breaking scalar potential

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}\left(D_\mu\Phi\right)^\dagger D^\mu\Phi + \alpha\Phi^\dagger\Phi - \beta\left(\Phi^\dagger\Phi\right)^2,$$

$$D_\mu\Phi = \left(\partial_\mu + ieA_\mu\right)\Phi, \quad \alpha,\ \beta > 0. \tag{6}$$

- Discovery of 't Hooft-Polyakov magnetic monopole as a soliton solution to $SU(2)$ non-Abelian Higgs (Georgi-Glashow) model (**'t Hooft, 1974; Polyakov, 1974**)
- Vortex of finite length require magnetic monopole at ends $\Longrightarrow$ magnetic charge confinement (**Nambu, 1974**)
- Nambu's magnetic confinement analogue $\Longrightarrow$
  quark confinement mechanism = dual superconductivity mechanism (**Mandelstam, 1976**):
  * Quarks emit chromoelectric field
  * Condensation of particles producing chromomagnetic field $\Longrightarrow$ QCD vacuum medium expels chromoelectric field (dual Meissner effect)
    $\Longrightarrow$ chromoelectric field flux form tube
    $\Longrightarrow$ Linear potential between quarks
    $\Longrightarrow$ Confinement

• Artificial magnetic monopole in QCD = **'t Hooft's Abelian projection**

- We need particles carrying chromo-magnetic charge in QCD to produce dual superconductivity
  * Presence of scalar field and scalar potential $\Longrightarrow$ existence of magnetic monopole;
  * However, no scalar field in QCD
- **Abelian projection (AP)**(**' t Hooft, 1981**):

– **Success of AP**

  * With AP, lattice simulation $\Longrightarrow$ condensation of artificial magnetic monopole and dual Meissner effect indeed happen

– **Shortcomings of AP**

  * Artificial magnetic monopole configuration has nothing with classical equation of motion
  * AP method cannot tell the physical meaning of confining configuration:
    classical field? Quantum field configuration?
  * Nothing helpful for analytical solution, lattice simulation has to be employed

The only way is resorting to supersymmetry and duality!!

2 4-D Supersymmetry and Feature of (Extended) Supersymmetric Gauge Theory

Supersymmetry

- Space-time symmetry with spin-1/2 fermionic generator $\Longrightarrow$

  Supermultiplet= fermions and bosons in one multiplet when supersymmetry classifies particles

- In a **supermultiplet**
  - members have equal mass and various internal degrees of freedom (electric charge, flavor, color, $\cdots$)
  - bosonic degrees of freedom (D.O.F)= fermionic D.O.F

- **Extended supersymmetry**
  - A number of ($N > 1$) sets of supersymmetries
  - **Global SUSY:** supermultiplet particle spin $\leq 1 \Longrightarrow N \leq 4$;
  - **Local SUSY:** gravity present, supermultiplet particle spin $\leq 2 \Longrightarrow N \leq 8$

- **Supersymmetry algebra** consist of
  - Poincaré symmetry
  - Fermionic symmetry

– Chiral R-symmetry $\Leftarrow$ Fermionic generator =Weyl spinor
$N = 1$: $U_R(1)$
$N = 2$: $SU_I(2) \times U_R(1)$
$N = 4$: $SU_R(4)$

Superspace and Superfield

- **Any symmetry transformation**, either space-time or internal, has **geometrical realization** = Translation or rotation of certain space;
  - Poincaré symmetry $\longleftrightarrow$ Minkowski space
  - Internal symmetry (electric charge, color, flavor, $\cdots$) $\longleftrightarrow$ abstract compact space
- **Supersymmetry** $\longleftrightarrow$ **Superspace**,

  e.g., $N = 1$ **case:**
  - space labeled by both C- and G(rassmann)-number coordinates $(x^\mu, \theta^\alpha, \overline{\theta}_{\dot{\alpha}})$
  - **(anti-)Chiral superspace**:
    Superspace labeled by $(x^\mu, (\overline{\theta}_{\dot{\alpha}})\theta^\alpha) \Longrightarrow$ Super-coordinates θ^α $(\overline{\theta}_{\dot{\alpha}})$ = left-(right-) Weyl spinors
- **Superfield** (e.g., $N = 1$ case:):
  - A field function $\Phi(x, \theta, \overline{\theta})$ in superspace

– Its polynomial expansion in terms of super-coordinate $\Longrightarrow$ supermultiplet;

e.g., the most general $N = 1$ **scalar superfield**

$$F(x,\theta,\overline{\theta}) = F(x) + \theta\eta(x) + \overline{\theta}\overline{\chi}(x) + \theta^2 M(x) + \overline{\theta}^2 N(x) + \theta\sigma^\mu\overline{\theta}A_\mu(x) + \theta^2\overline{\theta\lambda}(x) + \overline{\theta}^2\theta\eta(x) + \overline{\theta}^2\theta^2 D(x) \quad (7)$$

– $N = 1$ **(anti-)chiral superfield** $\Phi(x,(\overline{\theta})\theta)$**:**

* defined in chiral superspace
* obtained by imposing (anti-)chiral constraint on generic scalar superfield

$$(D_\alpha\Phi^\dagger = 0) \;\; \overline{D}_{\dot{\alpha}}\Phi(x,\theta,\overline{\theta}) = 0 \quad (8)$$

* its fermionic component is (right-)left-handed Weyl spinor
* give matter field supermultiplet (spin $s \leq \frac{1}{2}$)

– $N = 1$ **vector field** $V(x,\theta,\overline{\theta})$

* Obtained by reality constraint on generic superfield

$$V(x,\theta,\overline{\theta}) = V^\dagger(x,\theta,\overline{\theta}) \quad (9)$$

* give gauge field supermultiplet
* its fermionic component =Majorana spinor

– $N = 2$ **hypermultiplet**

* = One $N = 1$ chiral + one $N = 1$ anti-chiral superfield
* $N = 2$ matter field supermultiplet

– $N = 2$ **Chiral (vector) superfield**

  ∗ = One $N = 1$ vector + one $N = 1$ chiral superfield

  ∗ $N = 2$ gauge field supermultiplet

Supersymmetric Gauge Field Theory

1. **Classical construction:**

   Superrsymmetry impose strong constraint on dynamics:

   SUSY field theory action in superspace is always composed of **two types of terms**: e.g., $N = 1$ **case**

   - **D-type term:**
     – **integration over whole superspace**;
     $$\int d^4x d^2\theta d^2\overline{\theta}\, \overline{\Phi} e^{gV} \Phi \tag{10}$$
     – describes **kinetic terms of matter field supermultiplet; gauge interactions between matter and gauge supermultiplets**

   - **F-type term:**
     – **integration over (anti-)chiral superspace**
     $$\int d^4x \left[\left(\frac{1}{16\pi} \int d^2\theta \mathrm{Im}(\tau \mathrm{Tr} W^\alpha W_\alpha) + \int d^2\theta W[\Phi] \right) + (\mathrm{c.c.}) \right],$$
     $$\tau = \frac{\theta}{2\pi} + i\frac{4\pi}{g^2} \tag{11}$$

$W[\Phi]$ — Superpotential; its general form for a renormalizable action is

$$W[\Phi] = C\Phi + \frac{1}{2}m^2\Phi^2 + \frac{1}{3!}\lambda\Phi^3 \tag{12}$$

– give **Yang-Mills, gaugino terms, matter field mass terms, Yukawa terms, scalar potential etc.**

$N = 2$ **case:**

$N = 2$ **supersymmetry is more restrictive:**

- $N = 2$ **superspace and chiral (vector) superfield**

  – $N = 2$ superspace $(x^\mu, \theta, \overline{\theta}, \tilde{\theta}, \overline{\tilde{\theta}})$
  = introducing four more super-coordinates $\tilde{\theta}$ and $\overline{\tilde{\theta}}$ to $N = 1$ superspace $(x^\mu, \theta, \overline{\theta})$;

  – Generic $N = 2$ superfield $\Psi(x, \theta, \overline{\theta}, \tilde{\theta}, \overline{\tilde{\theta}})$
  =function in $N = 2$ superspace

  – Chirality and reality condition on generic $N = 2$ superfield

$$\begin{aligned} \Psi(x, \theta, \overline{\theta}, \tilde{\theta}, \overline{\tilde{\theta}}) &= \Psi^\dagger(x, \theta, \overline{\theta}, \tilde{\theta}, \overline{\tilde{\theta}}), \\ \overline{D}_{\dot{\alpha}}\Psi &= 0, \quad \overline{\tilde{D}}_{\dot{\tilde{\alpha}}}\Psi = 0, \end{aligned} \tag{13}$$

  $\Longrightarrow N = 2$ **chiral (also called vector) superfield**

$$\begin{aligned} \Psi &= \Phi(\tilde{y}, \theta) + \sqrt{2}\tilde{\theta}^\alpha W_\alpha(\tilde{y}, \theta) + \tilde{\theta}^\alpha\tilde{\theta}_\alpha F(\tilde{y}, \theta), \\ y^\mu &= x^\mu + i\theta\sigma^\mu\overline{\theta} + i\tilde{\theta}\sigma^\mu\overline{\tilde{\theta}}, \\ F(\tilde{y}, \theta) &= \Phi^\dagger(\tilde{y} - i\theta\sigma\overline{\theta}, \theta, \overline{\theta}) \exp\left[2gV(\tilde{y} - i\theta\sigma\overline{\theta}, \theta, \overline{\theta})\right]\Big|_{\overline{\theta\theta}} \end{aligned}$$

  $\Phi(\tilde{y}, \theta)$= $N = 1$ chiral superfield;
  $W_\alpha(\tilde{y}, \theta)$= $N = 1$ vector superfield strength

- **Supersymmetric Yang-Mills action** = Integration of $N=2$ prepotential $F[\Psi]$ over $N=2$ chiral superspace

$$S_{\rm YM} = \frac{1}{4\pi}\int d^4x {\rm ImTr}\int d^2\theta d^2\tilde{\theta}\frac{1}{2}\tau\Psi^2$$

(N=2 chiral superfield form)

$$= \frac{1}{8\pi}\int d^4x {\rm ImTr}\left[\tau\left(\int d^2\theta W^\alpha W_\alpha + 2\int d^2\theta d^2\overline{\theta}\overline{\Phi}e^{-V}\Phi\right)\right]$$

(N=1 chiral superfield form) (14)

- $N=2$ Matter field and gauge interaction
  - The squared mass of **massive** $N=2$ **matter field** must be equal to central charge $\Longleftarrow$ $N=2$ supersymmetry + spin ≤ 1
  - $N=2$ **matter field can only have (super-)gauge interaction** ($\Longleftarrow$ $N=2$ supersymmetry)

$$\begin{aligned} S_{\rm Matter} &= \int d^4x d^2\theta d^2\overline{\theta}\left(M^\dagger e^{-2gV}M + \widetilde{M}e^{2gV}\widetilde{M}^\dagger\right) \\ &+ \int d^4x\left[\int d^2\theta\left(\sqrt{2}\widetilde{M}\Phi M + m\widetilde{M}M\right) + {\rm c.c.}\right] \end{aligned} \quad (15)$$

2. **Quantum dynamical behavior constrained by supersymmetry:**

- **Global symmetry** $\Longrightarrow$

- conservative law at classical level (Nöther theorem)
- Ward identity among Green functions (**Building blocks of quantum effective action**) at quantum level = "selection rules "

  $\Longrightarrow$ Dominate quantum effective action

- Supersymmetric Ward identity is very powerful $\Longrightarrow$
  - No **perturbative** quantum correction to superpotential = celebrated **non-renormalization theorem**
  - Non-perturbative quantum correction to superpotential depend on field variables and parameters (mass, couplings, $\cdots$) **analytically**

$$W = W(\Phi, m, g), \quad \overline{W} = \overline{W}(\Phi^\dagger, m^\star, g^\star) \tag{16}$$

  $\Longrightarrow$ **Holomorphy of superpotential**

- **Holomorphic feature of SUSY guage theory is extremely important**

  $\Longrightarrow$ Non-perturbative (low-energy) **supersymmetric** gauge theory is more **easily tractable** than non-supersymmetric one

- **Anomaly supermultiplet**
  - Supersymmetry algebra $\Longrightarrow$
    conservative current supermultiplet, e.g., $N = 1$ case

$$(j_5^\mu, s^\mu, T^{\mu\nu}) \tag{17}$$

  j_5^μ — chiral $U_R(1)$ current;

s^μ — supersymmetry current
$T^{\mu\nu}$ — energy-momentum tensor

– Quantum correction violates some classical symmetries due to non-trivial topology of classical field configuration

$\Longrightarrow$ anomaly supermultiplet

$$\left(\partial_\mu j_5^\mu, \partial_\mu s^\mu, T^\mu_{\ \mu}\right) \tag{18}$$

$\Longrightarrow$ yield information on **anomalous quantum effective action**

- **Supersymmetric instanton calculus** (**Amit et al; Shifman etal; Affleck et al; 1980s**)
  – A big industry in non-perturbative SUSY gauge theory established in 1980s
  – Super-instanton preserve one-half of supersymmetry in Euclidean space
  – Super-instanton moduli space
  – · · ·

$\Longrightarrow$ A number of **non-perturbative results of SUSY gauge theory**

– **Supersymmetric 't Hoot effective action** constrained by holomorphy

– Celebrated Novikov-Shifman-Vainshtein-Zakharov **(NSVZ) β-function**

– Gaugino condensation and Konishi anomaly

– Insight into **dynamical supersymmetry breaking**

– lead to invention of **topological Yang-Mills theory** at the end 1980s —- **new type quantum field theory**

– ⋯

Solving low-energy $N = 2$ supersymmetric $SU(2)$ Yang-Mills theory in Coulomb phase

1. **Classical theory**

$$S_{\rm YM} = \frac{1}{4\pi}\int d^4x \int d^2\theta d^2\tilde{\theta}\,\frac{1}{2}{\rm ImTr}\left(\tau\Psi^2\right)$$

(N=2 chiral superfield form)

$$= \frac{1}{8\pi}\int d^4x {\rm ImTr}\left[\tau\left(\int d^2\theta W^\alpha W_\alpha + 2\int d^2\theta d^2\overline{\theta}\overline{\Phi}e^{-V}\Phi\right)\right]$$

(N=1 chiral superfield form)

$$= \frac{1}{g^2}\int d^4x {\rm Tr}\left(-\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + (D_\mu\phi)^\dagger D^\mu\phi - i\lambda\sigma^\mu D_\mu\overline{\lambda}\right.$$
$$\left. -i\overline{\psi}\overline{\sigma}^\mu D_\mu\psi - i\sqrt{2}[\lambda,\psi]\phi^\dagger - i\sqrt{2}[\overline{\lambda},\overline{\psi}]\phi - \frac{1}{2}[\phi^\dagger,\phi]^2\right)$$
$$+\frac{\theta}{16\pi^2}\int d^4x {\rm Tr}\left(F_{\mu\nu}\widetilde{F}^{\mu\nu}\right)$$

(2-component field form)

$$= \frac{1}{g^2}\int d^4x \Big(-\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} + (D^\mu\phi)^{a\star}(D_\mu\phi)^a + i\Psi^a\gamma^\mu(D_\mu\Psi)^a$$
$$+\frac{i}{\sqrt{2}}\epsilon^{abc}\overline{\Psi}^a[(1-\gamma_5)\phi^b + (1+\gamma_5)\phi^{b\star}]\Psi^c - \frac{1}{2}[\Phi,\Phi^\dagger]^{a2}\Big)$$
$$+\frac{\theta}{32\pi^2}\int d^4x F^a_{\mu\nu}\widetilde{F}^{a\mu\nu}$$

(4-component field form) (19)

2. **Generality on solving quantum field theory**

- Classical theory $\longrightarrow$ canonical quantization $\longrightarrow$ Schrödinger or Heisenberg equation $\longrightarrow$ Hilbert space
- Classical theory $\longrightarrow$ path integration quantization $\longrightarrow$ Quantum effective action
- **Wilsonian effective action**
  - Since Wilson's work on renormalization group flow $\Longrightarrow$ **Effective field theory supplant renormalization** $\longrightarrow$

    All the theories (except the ultimate theory?) are effective field theory and they are only valid to describe physics at certain energy scale

  - **Quantum effective action consists of three types of operators:**
    - **marginal:** mass dimension $= 4$;
    - **relevant:** mass dimension < 4, $\Lambda^n(\cdots)$, $n > 0$;
    - **irrelevant:** mass dimension > 4, $(\cdots)/\Lambda^n$, $n > 0$;

- For **massless SUSY gauge theory at low-energy**, only Wilsonian effective action is meaningful:
  - $*$ free of IR divergence (IR regularized)
  - $*$ holomorphic in superfields and parameters

3. **Vacuum ingredients** of $N = 2$ SUSY $SU(2)$ YM theory

   - Classical degenerate topological vacuum

     $\Longleftarrow H = 0$; classified by winding number from S^3 to $SU(2)$ (topological quantum number)

   - 't Hooft-Polyakov magnetic monopole and dyon $\Longleftarrow$ Presence of $SU(2)$ scalar field triplet and of Higgs vacuum $D_\mu \Phi = 0$ and $V(\Phi, \Phi^\dagger) = 0$

   - Yang-Mills instanton

   - Other exotic states: bound states from gaugino condensation, gaugino-monopole bound state, $\cdots$,

4. **Vacuum phase structure** of $N = 2$ SUSY $SU(2)$ YM theory

   - Confining phase
   - Higgs phase
   - Coulomb phase
   - $\cdots$

5. **Coulomb phase**

   It is the phase most easily to tackle

- **Arising of Coulomb phase**
  - **Unbroken supersymmetry** requires scalar potential vanishes in vacuum configuration

$$
\begin{aligned}
& V(\phi^\dagger,\phi) = \frac{1}{2}\left([\phi^\dagger,\phi]^a\right)^2 = 0 \\
\Longrightarrow\ & \langle\phi^a\rangle = v\sigma_3 \\
\Longrightarrow\ & SU(2) \longrightarrow U(1)
\end{aligned} \tag{20}
$$

  - **Super-Higgs effect** $\Longrightarrow$

    Original $N=2$ supremultiplet $= N=2$ massive $+\ N=2$ massless supermultiplets

  - **Integrating out massive modes and also massless quantum excitations above energy scale** Λ (Spare the physics with energy scale $\Lambda < v$)

    $\Longrightarrow$ Quantum dynamics of Coulomb phase descibed by Wisonian effective action

6. Heavy mode non-decoupling effects to Coulomb phase

- Fundamental field variables in Coulomb phase is $N=2$ $U(1)$ Abelian supermultiplet

$$
(\phi, \Psi^i, A_\mu) \tag{21}
$$

  classical dynamics looks trivial

- Quantum mechanically, heavy modes **do not decouple** completely

  $\Longrightarrow$ Perturbative quantum Correction in Coulomb phase

7. **Super-instanton Tunneling effects in Coulomb phase**

- Coulomb phase is quantum behavior of $N = 2$ non-Abelian $SU(2)$ Yang-Mills theory around vacuum $\phi^a = v\sigma_3$ and energy scale $\Lambda < v$;
- Instanton tunneling among degenerate topological vacuum configuration still exist

$\Longrightarrow$ **'t Hooft effective action** contributed by instanton tunneling amplitude among topological vacua but with **instant size** $< 1/v$

8. **BPS magnetic monopole and dyon in Coulomb phase**

- Considered as soliton at weak coupling (heavy, massive) chosen as vacuum state
- become light at strong coupling and behave like fundamental particle

$\Longrightarrow$ **Crucial point for electric-magnetic duality**

9. **Low-energy effective action**

- **Classical part** is $N = 2$ SUSY Abelian vector field theory (**free theory**)
- **Perturbative quantum correction part** from decoupling effects of heavy modes $\Longleftarrow$ **Two facts:**

  - Pertubative quantum correction of $N = 2$ SUSY gauge theory is one-loop exhausted
  - Anomaly supermultiplet fix it **completely**

- **Non-perturbative part**
  - Heritage from tunneling effects produced by super-instanton

- Sum up all available information:
  - $N = 2$ supersymmetry
  - Anomalous R-symmetry
  - 't Hooft effective action
  - Dimension analysis

$\Longrightarrow$ General form of **Wilsonian** quantum effective action for low-energy N=2 SUSY SU(2) YM theory **at weak coupling** in Coulomb phase (**Seiberg, 1988**)

$$\Gamma = \frac{1}{4\pi}\mathrm{Im}\int d^4x\ d^2\theta d^2\tilde{\theta}\mathcal{F}[\Psi]$$

(N=2 chiral superfield form)

$$= \frac{1}{4\pi}\mathrm{Im}\int d^4x \int d^2\theta\left[\frac{1}{2}\frac{\partial^2\mathcal{F}[\Phi]}{\partial\Phi^2}W^\alpha W_\alpha + \int d^2\theta d^2\overline{\theta}\overline{\Phi}\frac{\partial\mathcal{F}[\Phi]}{\partial\Phi}\right]$$

N=1 superfield form

$$
\mathcal{F}[\Psi] = \frac{i}{2\pi}\Psi^2 \ln \frac{\Psi^2}{\Lambda^2} + \Psi^2 \sum_{n=1}^{\infty} C_n \left(\frac{\Lambda^2}{\Psi^2}\right)^{2n}
$$

$$
\Psi = (\phi, \psi^i A_\mu) \tag{22}
$$

The last hurdle to exact solution: coefficient C_n

The resort: electric-magnetic duality

3 Electric-Magnetic Duality

1. **Duality in electromagnetism**

   - Electromagnetism in vacuum

$$
\begin{aligned}
&\partial_\nu F^{\mu\nu} = 0, \quad \partial_\nu {}^\star F^{\mu\nu} = 0, \\
&{}^\star F^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\lambda\rho} F_{\lambda\rho}, \quad \star^2 = -1 \\
&\Longrightarrow F^{\mu\nu} \longrightarrow {}^\star F^{\mu\nu}, \quad {}^\star F^{\mu\nu} \longrightarrow -F^{\mu\nu} \\
&\qquad \text{or} \quad (\mathbf{E}, \mathbf{B}) \longrightarrow (\mathbf{B}, -\mathbf{E})
\end{aligned} \tag{23}
$$

   $\Longrightarrow$ **electric-magnetic duality**

   - Electric source present, duality require the introduction of magnetic charge source

$$
\begin{aligned}
&\partial_\nu F^{\mu\nu} = j^\mu, \quad \partial_\nu {}^\star F^{\mu\nu} = k^\mu \\
&(F^{\mu\nu}, {}^\star F^{\mu\nu}) \longrightarrow ({}^\star F^{\mu\nu}, -F^{\mu\nu}), \\
&(j^\mu, k^\mu) \longrightarrow (k^\mu, -j^\mu) \\
&(e, g) \longrightarrow (g, -e)
\end{aligned} \tag{24}
$$

2. **Dirac monopole and quantization condition** (**Dirac, 1931**):

   - Violation of Bianchi identity (or source term for ${}^\star F^{\mu\nu}$)
     $\Longrightarrow A_\mu$ has singularity

   - Magnetic field produced by **Dirac monopole**

$$
\begin{aligned}
&\mathbf{B}(\mathbf{r}) = \nabla \times \mathbf{A} = \frac{\mathbf{g}}{\mathbf{4\pi}}\frac{\mathbf{r}}{\mathbf{r^3}} \\
&\Longrightarrow \mathbf{A}_\pm(\mathbf{r}) = \pm\frac{g}{4\pi r}\frac{1-\cos\theta}{\sin\theta}\hat{\mathbf{e}}_\phi
\end{aligned} \tag{25}
$$

on upper and lower hemisphere $S^2_\pm$;

$\theta = 0 \Longrightarrow z$-axis is **Dirac singular string**

- **Gauge transformation** on $S^2_+ \cap S^2_-$ (**Wu & Yang, 1975**)

$$\mathbf{A}_+ = \mathbf{A}_- + \nabla\chi(\phi), \quad \chi(\phi) = \frac{\mathbf{g}}{\mathbf{2}\pi}\phi,$$
$$g = \int_{S^2} \mathbf{B} \cdot d\mathbf{S} = \chi(2\pi) - \chi(0) \tag{26}$$

- **Dirac quantization condition**

Non-relativistic particle with electric charge $q = e$ in magnetic field produced by monopole

$$i\frac{\partial\psi}{\partial t} = -\frac{1}{2m}(\nabla + ieA)^2\psi \tag{27}$$

gauge invariance of Schrödinger equation (= single-valued wave function) $\Longrightarrow$

$$qg = 2\pi n, \quad n \in Z \tag{28}$$

- **Dirac-Zwanziger-Schwinger quantization condition**

Particle with both electric and magnetic charges (q_1, g_1) **(dyon)** in electric-magnetic field produced by dyon (q_2, g_2);

Conservation and quantization of angular momentum $\Longrightarrow$

$$q_1 g_2 - q_2 g_1 = 2\pi n, \quad n \in Z \tag{29}$$

$\Longrightarrow$ **Dual field theories describing electric and magnetic interaction must be strong/weak interchanged since e and g relate to electric and magnetic coupling**

3. **'t Hooft-Polyakov magnetic monopole and BPS magnetic monopole** ('t Hooft, 1974; Polyakov, 1974)

- **'t Hooft-Polyakov magnetic monopole**

  – more than 40 years later, **magnetic monopole** was found as a **topological soliton** in the bosonic part of Georgi-Glashow model

  – **Georgi-Glashow model** — $SO(3)$ (or $SU(2)$) gauge field couple with triplet scalar field model with scar potential potential implementing spontaneous symmetry breaking

$$\mathcal{L} = -\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} + \frac{1}{2}(D_\mu\phi)^a(D^\mu\phi)^a - \frac{1}{4}\lambda(\phi^2 - v^2)^2 \quad (30)$$

  – **Classical equation of motion:**

$$\begin{aligned} (D_\nu F^{\mu\nu})^a &= e\epsilon^{abc}\phi^b(D^\mu\phi)^c, \\ (D_\mu D^\mu\phi)^a &= -\lambda\phi^a(\phi^2 - v^2) \end{aligned} \quad (31)$$

  – **Ansatz (= gauge choice)** for monopole solution

$$\begin{aligned} \phi^a(\mathbf{r}) &= \frac{r^a}{er^2}H(evr), \\ A^a_i(\mathbf{r}) &= -\epsilon_{aij}\frac{r^j}{er^2}\left[1 - K(evr)\right], \\ A^a_0 &= 0, \\ a, i, j &= 1, 2, 3, \quad r = |\mathbf{r}| \end{aligned} \quad (32)$$

– **Boundary condition:**

$|\mathbf{r}| \to \infty$, field variables approch vacuum configuration

$$F^a_{\mu\nu} = 0, \quad (D_\mu\phi)^a = 0, \quad (\phi^a)^2 = v^2 \tag{33}$$

$\Longrightarrow$ Solution with finite energy (= **feature of soliton**)

$$\begin{aligned} E &= M_{\rm m} = \int d^3x \mathcal{H} \\ &= \int d^3x \left\{ \frac{1}{2} \left[(\Pi^a)^2 + (B^a_i)^2 + (E^a_i)^2 + \left[(D_i\phi)^a \right]^2 \right] \right. \\ & \quad \left. + \frac{\lambda}{4} \left(\phi^2 - v^2 \right) \right\} < \infty \\ E^a_i &= F^a_{i0}, \quad B^a_i = \frac{1}{2} \epsilon_{ijk} F^a_{jk} \end{aligned} \tag{34}$$

– **Asymptotic form of solution**

$$K(r) \sim \exp\left(-evr\right), \quad H(r) \sim \exp\left(-\sqrt{\lambda} vr\right) \tag{35}$$

– **Magnetic and topological feature:**

Boundary condition at large r:

$$(D_\mu\phi)^a = 0, \quad \phi^2 = v^2 \quad = \textbf{Higgs vacuum} \tag{36}$$

$\Longrightarrow SU(2) \longrightarrow U(1)$;

Electric-magnetic field = Projection of $SU(2)$ gauge field in the direction of Higgs field

$$A^a_\mu = \frac{1}{v^2 e} \epsilon^{abc} \phi^b \partial^\mu \phi^c + \frac{1}{v} \phi^a A^\mu,$$

$$
\begin{aligned}
F^a_{\mu\nu} &= \frac{1}{v}\phi^a F^{\mu\nu}, \\
F_{\mu\nu} &= \frac{1}{ev^3}\epsilon^{abc}\phi^a\partial_\mu\phi^b\partial_\nu\phi^c + \partial_\mu A_\nu - \partial_\nu A_\mu, \\
\partial_\nu F^{\mu\nu} &= 0, \quad \partial_\nu {}^\star F^{\mu\nu} = \frac{4\pi}{e}k^\mu, \\
k^\mu &= \frac{1}{8\pi v^3}\epsilon^{\mu\nu\lambda\rho}\epsilon^{abc}\partial_\nu\phi^a\partial_\lambda\phi^b\partial_\rho\phi^c \qquad (37)
\end{aligned}
$$

$\Longrightarrow$ **magnetic charge**

$$
\begin{aligned}
g &= \frac{4\pi}{e}\int d^3x k^0 = \frac{4\pi}{e}n_{\rm m} \\
\Longrightarrow \quad eg &= 4n_{\rm m}\pi, \quad n_{\rm m} \in Z \qquad (38)
\end{aligned}
$$

$\Longrightarrow$

* Classical solution presents feature of magnetic monopole;
* It becomes Dirac monopole at large distance ($n = 2$**,** $n_{\rm m} = 1$ **for above ansatz**);
* magnetic charge is topological quantum number = winding number for the mapping: $S^3 \to SU(2)$

- **Dyon** (**Julia & Zee, 1975**)

  Soliton solution with both electric charge and magnetic charge

- **BPS monopole (dyon) and BPS bound** (**Bogomol'nyi, 1976; Prasad and Sommerfield, 1975**)

  – It is 't Hooft-Polyakov monopole (or dyon) with **minimal**

mass for given magnetic (and electric) charge

$$M_{\rm m} \geq \frac{1}{2}\int d^3x \left[(E_i^a)^2 + (B_i^a)^2 + [(D_i\phi^a)]^2\right]$$

$$\begin{aligned} = &\ \frac{1}{2}\int d^3x \left(|E_i^a - (D_i\phi)^a \sin\theta|^2 + |B_i^a - (D_i\phi)^a\cos\theta|^2\right) \\ &+ \sin\theta \int d^3x E_i^a (D_i\phi)^a + \cos\theta\int d^3x B_i^a(D_i\phi)^a \\ \geq &\ \sin\theta\int d^3x E_i^a(D_i\phi)^a + \cos\theta\int d^3x B_i^a (D_i\phi)^a \\ \Longrightarrow &\ M_{\rm m} \geq v\sqrt{e^2+g^2} = v|e+ig| \end{aligned} \quad (39)$$

– **Saturation of Bogomol'nyi bound for monopole** when

  * $V(\phi) = 0 \Longleftarrow$ Take $\lambda \to 0$ but keep boundary condition $\phi^2 = v^2$ (**Prasad-Sommerfield limit**)

– In particular, when BPS bound is saturated, $\lambda = 0$ and $\int d^3x k_0 = 1$,

$\Longrightarrow$ Classical equation of motion (31) is equivalent to **Bogomol'nyi equation** ($\theta = 0$)

$$B_i^a = \pm(D_i\phi)^a \quad (40)$$

$\Longrightarrow$ **analytical solution = BPS monopole**

4. **Montonen-Olive duality** (**Montonen and Olive, 1977**)

- **Higgs mechanism in Georgi-Glashow model**

$$V(\phi) = 0 \Longrightarrow SU(2) \longrightarrow U(1),$$
$$\Longrightarrow A^a_\mu, \quad \phi^a \longrightarrow W^\pm_\mu, A_\mu, H \tag{41}$$

- **Comparison between fundamental particle and BPS monopole spectrums** $(\lambda \to 0)$

Particle	Mass	Electric charge	Magnetic charge	Spin/ Helicity
Higgs	0	0	0	0
Photon	0	0	0	± 1
$W^\pm_\mu$	ev	$\pm e$	0	1
$M_\pm$	gv	0	$\pm g$	0

$\Longrightarrow$

- Both fundamental and solitonic particles satisfy BPS bound

$$M_{\rm m} = v\sqrt{e^2 + g^2} = v|e + ig| \tag{42}$$

- Electric-magnetic Z_2 duality $\Longleftarrow$ BPS particle spectrum is symmetric under
  - $(e, g) \longleftrightarrow (g, -e)$
  - BPS monopole $M_\pm \longleftrightarrow$ Massive vector bosons $W^\pm_\mu$

  $\Longrightarrow$

- **Montonen-Olive duality conjecture:**

There should exist a dual (magnetic) description to Georgi-Glashow model where

– elementary gauge particles are BPS monopole

– they form $SU(2)$ gauge triplet with photon

– vector bosons $W_\mu^\pm$ behave as " electric monopole "

– gauge coupling is opposite to that in electric theory ($\Longleftarrow$ Dirac quantization condition)

$\Longrightarrow$ **Start of modern duality**

Remark: Bosonization —- another type of duality proposed in 1975 (**Coleman, 1975; Mandelstam, 1975**) $\Longrightarrow$

sine-Gordon theory = massive Thirring model

- **Physical support to M-O duality**

  – long-range force between $W_\mu^\pm$ = that between BPS magnetic monopole

- **Drawbacks in Montonen-Olive duality conjecture**

  (a) Coleman-Weinberg mechanism
  $\Longrightarrow$ quantum correction to scalar potential
  $\Longrightarrow$ **modification of classical mass spectrum and of Bogomol'nyi bound by quantum correction**

  (b) Quantum correction in gauge theory
  $\Longrightarrow$ Running of gauge couplings
  $\Longrightarrow$ **violation of Dirac quantization condition and hence duality**

(c) **Spin unmatch of dual particles**

– $W_\mu^\pm$ have spin 1;

– BPS monopole is spherically symmetric and spin=0

(d) **No way to prove the conjecture**

– one is in weak coupling

– the other is strongly coupled

$\Longrightarrow N = 4$ **supersymmetric Yang-Mills theory get rid of almost all these shortcomings!!**

5. **Witten Effect and $SL(2,Z)$ electric-magnetic duality**

- Instanton tunneling effect in Georgi-Glashow model $\Longrightarrow$

  – Lifting of degenerate topological vacua

  – Correct vacuum = θ **vacuum**

  It is linear combination of topological vacua $|N\rangle$

$$|\theta\rangle = \sum_N e^{iN\theta}|N\rangle, \quad N \in Z \tag{43}$$

– The vacuum to vacuum transition amplitude

$$\begin{aligned} Z &= \lim_{T\to\infty} \langle \theta, t_i = -T/2|\theta, t_f = T/2\rangle \\ &= \lim_{T\to\infty} \langle \theta| \exp(-HT)|\theta\rangle \\ &\sim \int \mathcal{D}A\, \delta[A_0^a] \exp\left[-\int d^3x\, (\mathcal{L} + \mathcal{L}_\theta)\right] \end{aligned}$$

$$\mathcal{L} = -\frac{1}{4e^2}F^a_{\mu\nu}F^{a\mu\nu}, \quad \mathcal{L}_\theta = \frac{\theta}{32\pi^2}F_{a\mu\nu}{}^\star F^{a\mu\nu} \tag{44}$$

$\Longrightarrow$ **Georgi-Glashow model with θ (strong CP violation) term**

$$\begin{aligned}\mathcal{L} + \mathcal{L}_\theta &= -\frac{1}{16\pi}\mathrm{Im}\left[\tau\left(F^a_{\mu\nu}F^{a\mu\nu} - iF^a_{\mu\nu}{}^\star F^{a\mu\nu}\right)\right] \\ &\quad +\frac{1}{2e^2}(D_\mu\phi)^a(D^\mu\phi)^a - V(\phi), \\ \tau &= \frac{\theta}{2\pi} + i\frac{4\pi}{e^2}\end{aligned} \tag{45}$$

- **Witten effect:** (**Witten, 1979**)

  Due to θ term, electric charge of a dyon receive extra contribution proportional to its magnetic charge

  – Gauge transformation in the direction of Higgs field
  = **Global $U(1)$ transformation**

  Nöther theorem $\Longrightarrow$ electric charge

$$\delta A^a_\mu = -\frac{1}{e}(D_\mu\Lambda)^a = -\frac{1}{ea}(D_\mu\phi)^a;$$

$$\begin{aligned}Q_e &= \int d^3x\, j^a_0 \\ &= \int d^3x\, \delta A^a_i\, \frac{\delta}{\delta(\partial_0 A^a_i)}(\mathcal{L}+\mathcal{L}_\theta) \\ &= -\frac{1}{ea}\int d^3x\left(E^a_i(D_i\phi)^a - \frac{\theta e^2}{8\pi^2}B^a_i\right)(D_i\phi)^a\end{aligned}$$

$$= -\frac{1}{e}\left(q - \frac{\theta e^2}{8\pi^2}\right) \tag{46}$$

– **Finite** $U(1)$ gauge transformation invariance $\Longrightarrow$

$$\begin{aligned} &\exp(2\pi i Q_e) = 1 \\ &\Longrightarrow Q_e = -n_e \in Z \\ &\Longrightarrow q = n_{\rm e} e + \frac{e\theta}{2\pi} n_{\rm m} \end{aligned} \tag{47}$$

- **Witten effect** $\Longrightarrow$ Z_2 **duality promote to** $SL(2,Z)$ **duality**

  Witten effect has great impact to Montonen-Olive duality and play crucial role in later development $\Longrightarrow$ **complex parameter** τ **enter the theory**

  Periodic property of θ **and electric-magnetic duality**

  $1/32\pi^2 \int d^4x F_{a\mu\nu}{}^{\star}F^{a\mu\nu} \in Z \Longrightarrow$ " large " gauge symmetry of theory = periodicity of θ $\Longrightarrow$

  – **Coupling:**

$$\begin{aligned} \theta \longrightarrow \theta + 2\pi \;&\Longleftrightarrow\; T : \tau \longrightarrow \tau + 1 \\ e \longrightarrow g = -\frac{4\pi}{e} \;&\Longleftrightarrow\; S : \tau \longrightarrow -\frac{1}{\tau} \end{aligned} \tag{48}$$

and

$$\begin{aligned} \mathrm{Im}\,\tau = \frac{4\pi}{e^2} > 0, \\ S^2 = 1, \quad (ST)^3 = 1 \end{aligned} \tag{49}$$

$\Longrightarrow S$**,** T **generate discrete** $SL(2,Z)$ **group**

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \tau = \frac{a\tau + b}{c\tau + d},$$
$$ab - cd = 1, \quad a,b,c,d \in Z \tag{50}$$

– Invariance of BPS mass spectrum under $SL(2,Z)$

$$g = n_m \frac{4\pi}{e}, \quad q = n_e e + n_m \frac{e\theta}{2\pi},$$

$$\Longrightarrow \quad M_{\rm m}^2 = v^2(q^2 + g^2)$$
$$= 4\pi v^2 \mathbf{n}^T \cdot A(\tau) \cdot \mathbf{n},$$

$$\mathbf{n} = \begin{pmatrix} n_e \\ n_m \end{pmatrix}, \quad n_e, n_m \in Z$$

$$A(\tau) = \frac{1}{{\rm Im}\tau} \begin{pmatrix} 1 & {\rm Re}\tau \\ {\rm Re}\tau & |\tau|^2 \end{pmatrix}, \tag{51}$$

Under $SL(2,Z)$ **transformation**

$$\begin{aligned} \mathbf{n} &\longrightarrow L \cdot \mathbf{n}, \qquad L \in SL(2,Z), \\ A(L\cdot\tau) &= (L^{-1})^T \cdot A(\tau) \cdot L^{-1}, \\ M_{\rm m}^2 &= \text{invariant} \end{aligned} \tag{52}$$

These two evidences $\Longrightarrow$ bolder conjecture:

Montonen-Olive Z_2 **duality promote to** $SL(2,Z)$ **duality**

6. **Montonen-Olive-Osborn duality in $N = 4$ supersymmetric Yang-Mills theory** (**Osborn , 1979**)

Montonen-Olive can only possibly exist in $N = 4$ SUSY Y-M theory (maximal global supersymmetry)

- **Extended $N = 2, 4$ supersymmetry in BPS magnetic monopole background**
  - **Witten-Olive's recognition on the physical meaning of central charge in supersymmetry algebra (Witte, and Olive, 1978) :**
    * Symmetry algebra of a field theory with **topologically non-trivial field configuration** can admit central charge
    * **Central charges in extended supersymmetry algebra = electric and magnetic charges of dyonic states**
    * **Saturation of supersymmetry bound = BPS bound!!**

  On one hand, **extended supersymmetry algebra**

$$\left\{Q^i_\alpha, \overline{Q}^j_\beta\right\} = \delta^{ij} \left(\gamma^\mu\right)_{\alpha\beta} P_\mu + \delta_{\alpha\beta} U^{ij} + i \left(\gamma_5\right)_{\alpha\beta} V^{ij}$$

e.g. $N = 2, \quad U_{ij} = \epsilon_{ij} U, \quad V_{ij} = \epsilon_{ij} V,$

In rest frame, $\quad P_\mu = (M, 0, 0, 0)$

Two-component form:

$$\left\{Q^i_\alpha, \overline{Q}^j_{\dot\alpha}\right\} = 2M\delta^{ij}\delta_{\alpha\dot\alpha},$$
$$\left\{Q^i_\alpha, Q^j_\beta\right\} = 2\epsilon^{ij}\epsilon_{\alpha\beta}(U+iV),$$
$$\left\{\overline{Q}^i_{\dot\alpha}, \overline{Q}^j_{\dot\beta}\right\} = 2\epsilon^{ij}\epsilon_{\dot\alpha\dot\beta}(U-iV),$$

Combination:

$$a_\alpha = Q^1_\alpha + \epsilon_{\alpha\beta}\overline{Q}_{2\dot\beta}\overline{\sigma}^{0\dot\beta\beta},$$
$$(a_\alpha)^\dagger = \overline{Q}_{1\dot\alpha} + \epsilon_{\dot\alpha\dot\beta}\overline{\sigma}^{0\dot\beta\beta}\overline{Q}^2_\beta,$$
$$b_\alpha = Q^1_\alpha - \epsilon_{\alpha\beta}\overline{Q}_{2\dot\beta}\overline{\sigma}^{0\dot\beta\beta},$$
$$(b_\alpha)^\dagger = \overline{Q}_{1\dot\alpha} - \epsilon_{\dot\alpha\dot\beta}\overline{\sigma}^{0\dot\beta\beta}\overline{Q}^2_\beta, \Longrightarrow$$

$$\left\{a_\alpha, (a_\beta)^\dagger\right\} = 2(M_{\mathrm{m}} + |Z|)\sigma^0_{\alpha\dot\beta},$$
$$\left\{b_\alpha, (b_\beta)^\dagger\right\} = 2(M_{\mathrm{m}} - |Z|)\sigma^0_{\alpha\dot\beta},$$
$$Z = U + iV \tag{53}$$

On the other hand, **in field theory realization**

For $N = 2$ **SUSY YM theory**,

$$\phi^a = P^a + iS^a, \quad \langle P^a\rangle = v, \quad \langle S^a\rangle = 0,$$
$$U = \int d^3x\partial_i(P^aE^a_i + S^aB^a_i) \sim qv,$$
$$V = \int d^3x\partial_i(P^aB^a_i + S^aE^a_i) \sim gv \tag{54}$$

(53) and (54) $\Longrightarrow$ Bound $M^2_{\mathrm{m}} \geq v^2(q^2+g^2)$

Saturation of the bound $\Longrightarrow$

– BPS monopole (or dyon) **break (or preserve) one-half of supersymmetries**

- BPS monopole (or dyon) belong to **short massive supermutiplet representation** of extended supersymmetry (2^N rather than 2^{2N} states)

- **Classical mass formula** for BPS dyon and monopole are **protected by supersymmetry** and **not spoiled by quantum correction**

- **Alternative approach:**

  Straightforward check on supersymmetry transformation in monopole background $\Longrightarrow$ one-half of SUSYs break (or preserve)

Evidence for M-O duality in $N = 4$ SUSY YM theory:

- Non-renormalization theorem from SUSY
  $\Longrightarrow$ No quantum correction to scalar potential
  $\Longrightarrow$ **Classical mass formula for fundamental particle stands**

- **Mass formula** of BPS magnetic monopole is preserved by SUSY

- β-function =0, $\Longrightarrow$ it is a conformal field theory, **no running of gauge coupling**

- **Spin match** between fundamental particle and BPS magnetic monopole supermultiplets
  - $N = 4$ **BPS monople** belong to short representation of SUSY algebra: 1 (1); 4 (1/2); 5 (0)

Witten and Seiberg used **electric-magnetic duality conjecture in low-energy** $N = 2$ **SUSY gauge theory** to get the exact solution
$\longrightarrow$ **It is not Montonen-Olive duality !!**

4 Seiberg-Witten Theory

Main idea (Seiberg and Witten, 1994):

Low-energy effective action at weak-coupling
+ electric-magnetic duality conjecture

$\Longrightarrow$ Global structure of vacuum moduli space of N=2 SUSY YM theory = 1-D complex manifold whose Riemannian surface is torus

$\Longrightarrow$ **Physical problem converted into 1-D complex geometrical problem**

$\Longrightarrow$ Exact solution

Main steps and techniques:

1. From classical scalar potential, determine **physical quantity describing classical vacuum moduli space** of $N = 2$ SUSY $SU(2)$ YM theory in **Coulomb phase**

   - Supersymmetry requirement on vacuum configuration $\Longrightarrow$

$$
\begin{aligned}
& V(\phi) = \frac{1}{2e^2}\mathrm{Tr}[\phi, \phi^\dagger]^2 = 0 \\
& \Longrightarrow \text{classical vacuum configuration:} \;\; \phi = a\sigma_3 \\
& \Longrightarrow SU(2) \longrightarrow U(1) \;\; \text{when} \;\; a \neq 0
\end{aligned}
\tag{55}
$$

   - Gauge symmetry requires that **appropriate quantity characterizing vacuum configuration** is

$$
u(a) = \frac{1}{2}\mathrm{Tr}\phi^2 = a^2 \tag{56}
$$

2. Analysis on the **existence of electric-magnetic duality** in Coulomb phase of $N = 2$ SUSY $SU(2)$ YM theory based on general form of its Wilsonian effective action (22) $\Longrightarrow$

   - Magnetic dual theory should be $N = 2$ **supersymmetric QED theory with magnetic monopole as hypermultiplet** (**N=2 matter**)

     In $N = 1$ superfield,

$$
\begin{aligned}
A, W_\alpha &\longleftrightarrow A_D, W_{D\alpha}, M, \widetilde{M} \\
\tau(A) &\longleftrightarrow \tau(A_D),
\end{aligned}
\tag{57}
$$

   - The couplings of dual theories are opposite

$$
\begin{aligned}
\tau(A_D) &= -\frac{1}{\tau(A)}, \\
\tau(A) &= \frac{\partial^2 \mathcal{F}[A]}{\partial^2 A}
\end{aligned}
\tag{58}
$$

   - Writing dual theory in $N = 1$ superfield form, **relation between the dual $N = 1$ chiral superfields in $N = 2$ chiral (vector) supermultiplet** is

$$
A_D = \frac{\partial \mathcal{F}[A]}{\partial A}, \quad A = -\frac{\partial \mathcal{F}[A_D]}{\partial A_D}
\tag{59}
$$

   - Consider low-energy quantum effective action as **sigma model** $\Longrightarrow$

     Imaginary part of complexified effective gauge coupling $\tau(a)$ **= metric of vacuum moduli space**

$$
\tau(a) = \tau\left(\langle A \rangle|_{\theta=\bar{\theta}=0} = a\right) = \frac{\theta(a)}{2\pi} + \frac{4\pi i}{g^2(a)},
$$

$$
\begin{aligned}
ds^2 &= \mathrm{Im}\tau(a) da d\bar{a} = \left(\mathrm{Im}\frac{\partial^2\mathcal{F}[a]}{\partial^2 a}\right) da d\bar{a} \\
&= \left(\mathrm{Im}\frac{\partial a_D}{\partial a}\right) da d\bar{a} = \mathrm{Im}\, da_D d\bar{a} \\
&= -\frac{i}{2}\left(da_D d\bar{a} - da d\bar{a}_D\right) = \mathrm{Im}\frac{da_D}{du}\frac{d\bar{a}}{d\bar{u}} du d\bar{u}, \\
\tau(a) &= \frac{\partial^2\mathcal{F}[a]}{\partial^2 a} = \frac{da_D}{da} = \frac{da_D(u)/du}{da(u)/du} \qquad (60)
\end{aligned}
$$

$\Longrightarrow a$ and a_D = coordinates of moduli space

- **Physical meaning of a_D and a:**

$$
\begin{aligned}
a\sigma_3 &= \langle\phi\rangle = \langle A\rangle|_{\theta=\bar{\theta}=0}\,, \\
a_D\sigma_3 &= \langle\phi_D\rangle = \langle A_D\rangle|_{\theta=\bar{\theta}=0} \qquad (61)
\end{aligned}
$$

- Then **quantum moduli space** can be tackled by analyzing its geometrical structure $\Longrightarrow$
  - **Electric-magnetic duality transformation = monodromies —— crucial observation by Seiberg and Witten**
    - ∗ Monodromies —- coordinate transformation in moduli space obtained by encircling singular points on moduli space

3. **Analysis in weak-coupling region**
(="semi-classical" analysis
= perturbative analysis
=$u \to \infty$ **region analysis**)

- Weak-coupling region, perturbation theory works $\Longrightarrow$

  one-loop $N=2$ prepotential $\mathcal{F}_{\rm one-loop}$ gives **monodromy at singularity** $u=\infty$

$$\mathcal{F}_{\rm one-loop} = \frac{i}{2\pi}A^2 \ln\frac{A^2}{\Lambda^2}, \quad a^2 \approx u,$$
$$a_D = \frac{ia}{\pi} + \frac{2ia}{\pi}\ln\frac{a}{\Lambda} \tag{62}$$

  Make a close loop on u-plane around $u=\infty$

$$\ln u \longrightarrow \ln u + 2\pi i, \quad \begin{pmatrix} a_D \\ a \end{pmatrix} \longrightarrow M_\infty \begin{pmatrix} a_D \\ a \end{pmatrix},$$
$$M_\infty = \begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix} \tag{63}$$

4. **Strong-coupling region can be reached through complex geometry analysis and physical interpretation is provided by electric-magnetic duality**

- According to complex analysis, for a multi-valued complex function, one singularity $\Longrightarrow$

  The existence of **at least one more singularity at finite** u

  $\Longrightarrow$ This push the theory to **strong coupling region**

- Z_2 symmetry spared from the anomalous $U_R(1)$ symmetry and positive definiteness of $\mathrm{Im}\tau(a)$

$\Longrightarrow$ Prepotential should have **only two singularities** at

$$u = \pm\Lambda^2 \tag{64}$$

- **Analysis on the physical meaning of two finite singularites and dynamical behavior around them**

  – Physical analysis enlightened from 't Hooft's argument on confinement and impossibility for the existence of massless "electric" gauge boson

  $\Longrightarrow$ **Singularities should only be massless magnetic monopole or dyon**

  – Electric-magnetic duality $\Longrightarrow$

  The dual of strongly coupled "electric " theory = weakly coupled $N = 2$ SUSY QED with magnetic monopole as hypermultiplet

  $\Longrightarrow$ one-loop beta function of dual magnetic theory

$$\beta(g_D) = \frac{g_D^2}{8\pi}, \quad \text{Im}\tau_D = \frac{4\pi}{g_D^2(\Lambda)},$$
$$a_D \frac{d}{da_D}\tau_D = -\frac{i}{\pi},$$
$$\Longrightarrow \tau_D = -\frac{i}{\pi} \ln\, a_D = -\frac{da}{da_D}$$
$$\Longrightarrow a = a_0 + c_0 \frac{i}{\pi} a_D \ln\, a_D - \frac{i}{\pi} c_0 a_D \tag{65}$$

$a_D(\Lambda^2) = 0$ **around** $u = \Lambda^2$ **means** $\Longrightarrow$

$$a_D = c_0(u - \Lambda^2),$$
$$a = a_0 + \frac{i}{\pi} c_0 (u - k^2) \ln (u - \Lambda^2) - \frac{i}{\pi} c_0 \ln(u - \Lambda^2)$$

when $(u - \Lambda^2) \longrightarrow e^{2i\pi}(u - \Lambda^2)$,
$\Longrightarrow$ Monodromy at singularity $u = \Lambda^2$

$$M_{\Lambda^2} = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} \tag{66}$$

– Determine the monodromy at the singularity $u = -\Lambda^2$

∗ Using complex analysis $\Longrightarrow$
a contour around $u = \infty$ **deform into two loop around** $u = \pm\Lambda^2$ with same base point
$\Longrightarrow$

$$M_\infty = M_{\Lambda^2} M_{-\Lambda^2} \Longrightarrow M_{-\Lambda^2} = \begin{pmatrix} -1 & 2 \\ -2 & 3 \end{pmatrix} \tag{67}$$

– Using **invariance of BPS mass spectrum under** $SL(2, Z)$ **duality**

$$m^2 = |Z|^2, \quad Z = (n_m, n_e) \begin{pmatrix} a_D \\ a \end{pmatrix},$$
$$\begin{pmatrix} a_D \\ a \end{pmatrix} \to M \begin{pmatrix} a_D \\ a \end{pmatrix},$$
$$(n_m, n_e) \to (n_m, n_e) M^{-1} \tag{68}$$

– If $m^2 = 0$, the requirement

$$(n_m, n_e)M^{-1} = (n_m, n_e) \tag{69}$$

and normalization $\Longrightarrow$

– **Physical meaning of two finite Singularities:**

(a) at $u = \Lambda^2$ = monopole with charge

$$(n_m, n_e) = (1, 0) \tag{70}$$

(b) at $u = -\Lambda^2$ = dyon with charge

$$(n_m, n_e) = (1, -1). \tag{71}$$

5. **Global structure of moduli space and exact solution**

- Once monodromies on a complex plane have been figured out, solving theory convert in to a type problem in complex analysis

  = celebrated **Riemann-Hilbert problem:**

  Look for multi-valued complex functions $a(u)$ and $a_D(u)$ that have monodromies $M_{\infty,\pm\Lambda^2}$ around singularities ∞, $\pm\Lambda^2$ on complex u-plane

- **Two approaches:**

  – $a(u)$ and $a_D(u)$ = Periodic integral in the Riemann surface of u-plane = Torus (**Seiberg and Witten , 1994**)

  – $a(u)$ and $a_D(u)$ = Solution of differential equation (**Picard-Fuchs equation**) with regular singular points

- **First approach:**
  - Any two of $M_{\pm\Lambda^2,\infty}$ generate discrete subgroup $\Gamma_0(2) \subset SL(2,Z)$

$$\Gamma_0(2) \;=\; T \in SL(2,Z),$$

$$T \;=\; \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad b = 2n \tag{72}$$

$\Longrightarrow$ **The quantum moduli space**

$$\mathcal{M}_q = \mathcal{H}_+/\Gamma_0(2) \tag{73}$$

$\mathcal{H}_+$ — upper half u-plane
$\Longrightarrow$

  - **$\mathcal{M}_q$ has a family of algebraic curve representation parameterized by u**

$$y^2(x,u) = (x-\Lambda^2)(x+\Lambda^2)(x-u), \quad x \in C \tag{74}$$

$\Longrightarrow$

  - **$\mathcal{M}_q$ = torus with $\tau(u)$ as moduli parameter**

  Argument for this statement:

    * y is multi-valued on complex x-space

$$y(x,u) = \pm\sqrt{(x-\Lambda^2)(x+\Lambda^2)(x-u)} \tag{75}$$

$\Longrightarrow$ four branch points $(-\Lambda^2, \Lambda^2, u, \infty)$.

* Riemann surface on which y is single-valued is defined along branch cuts

$$[-\Lambda^2, \Lambda^2], \quad [u, \infty) \tag{76}$$

$\Longrightarrow$

– **Solution to $a(u)$ and $a_D(u)$**

Effective complex coupling :

$$\tau(u) = \frac{da_D}{da} = \frac{\Omega_D(u)}{\Omega(u)}$$
$$\Omega(u) = \oint_\alpha \omega = \frac{\partial a(u)}{\partial u}, \quad \Omega_D(u) = \oint_\beta \omega = \frac{\partial a_D(u)}{\partial u}$$
$$\omega = \frac{1}{\sqrt{2}\pi} \frac{dx}{y(x,u)} = \text{holomorphic differential} \tag{77}$$

α, β =canonical basis for homological cycles of torus

$\Longrightarrow$ We finally get

$$\begin{aligned} a(u) &= \frac{\sqrt{2}}{\pi} \int_{-1}^{1} dx \, \frac{\sqrt{x-u}}{\sqrt{x^2-\Lambda^4}}, \\ a_D(u) &= \frac{\sqrt{2}}{\pi} \int_{1}^{u} dx \, \frac{\sqrt{x-u}}{\sqrt{x^2-\Lambda^4}}, \end{aligned} \tag{78}$$

– **Once a and a_D is known**

$$a_D = \frac{\partial \mathcal{F}(a)}{\partial a}, \quad a = -\frac{\partial \mathcal{F}_D(a_D)}{\partial a_D}, \tag{79}$$

$\Longrightarrow$ **The** $N = 2$ **prepotential:**

(a) At the region $u = \infty$, the appropriate variable expressing prepotential is a

$$\mathcal{F}(a) = \frac{ia^2}{2\pi} \ln\left(\frac{a^2}{\Lambda^2}\right) + \frac{a^2}{2i\pi} \sum_{n=1}^{\infty} C_n \left(\frac{a^2}{\Lambda^2}\right)^{2n} \tag{80}$$

(b) At the region $u = \Lambda^2$, **strong coupling electric theory = weak-coupling magnetic theory**, the appropriate variable describing prepotential is a_D

$$\mathcal{F}_D(a_D) = -\frac{ia_D^2}{8\pi} \ln\left(\frac{a_D^2}{\Lambda^2}\right) - \frac{\Lambda^2}{2i\pi} \sum_{n=1}^{\infty} C_n^D \left(\frac{ia_D}{\Lambda}\right)^{n} \tag{81}$$

(c) At the region $u = -\Lambda^2$, using monodromy, the appropriate variable describing prepotential is $\tilde{a}_D = 2a_D - a$

$$\widetilde{\mathcal{F}}_D(\tilde{a}_D) = \mathcal{F}_D(2a_D - a) \tag{82}$$

$\Longrightarrow$ **Exact solution**:

coefficients C_n and C_n^D can be extracted out from $a(u)$ and $a_D(u)$.

6. **Second approach: explicit expression on solution from differential equation**

5 (Abelian) Confinement in N=1 SUSY Yang-Mills theory from magnetic monopole condensation

Seiberg-Witten theory $\Longrightarrow$
quantitative mechanism for confinement in N=1 $SU(2)$ SUSY YM theory

1. Feature of $N = 1$ $SU(N_c = 2)$ SUSY pure YM theory:

   - **Classical action**

$$\begin{aligned}\mathcal{L}_{\rm SYM} &= -\frac{1}{4g^2}F^a_{\mu\nu}F^{a\mu\nu} + \frac{1}{g^2}\overline{\lambda}^a i\not{D}\lambda^a + \frac{i\theta}{32\pi^2}F^a_{\mu\nu}\widetilde{F}^{a\mu\nu} \\ &= \frac{1}{16\pi}\int d^2\theta\,{\rm Im}\,(\tau W^{a\alpha}W^a_\alpha) \end{aligned} \quad (83)$$

   - **Dynamical behavior**

     Like ordinary QCD in some aspects, but **fermions in adjoint representation of gauge group and presence of supersymmetry**

     – Asymptotic freedom at high energy and Color confinement at low-energy

     – Gluinos condense into vacuum in pairs and hence " chiral " symmetry breaking

     – **Mass gap:** non-vanishing anomaly supermultiplet + gluino condensation $\Longrightarrow$ $N = 1$ low-energy SUSY YM theory has **no massless particles** at low-energy (**Veneziano & Yankielowicz, 1982**)

- Vacuum structure
  - Number of vacua in SUSY gauge theory can be counted from Witten index $\mathrm{Tr}(-1)^F$
  - $\Longrightarrow N = 1$ $SU(N_c = 2)$ SUSY YM theory has two ($N_c = 2$) discrete vacua connected by Z_2 symmetry spared by $U_R(1)$-symmetry
  - These two vacua are characterized by gaugino condensation $\langle\lambda\lambda(x)\rangle$

2. Relation between $N = 1$ and $N = 2$ SUSY $SU(N_c = 2)$ Yang-Mills theory (**Seiberg & Witten, 1994**)

- $N = 1$ $SU(2)$ **Yang-Mills theory = Low-energy effective theory of** $N = 2$ **SU(2) Yang-Mills theory**
  - Add a soft supersymmetry breaking term

$$W_{\mathrm{m}} = m\mathrm{Tr}\Phi^2 \tag{84}$$

  in $N = 1$ chiral superfield sector of $N = 2$ SUSY YM theory **in Coulomb phase** $\Longrightarrow$
    - ∗ lift the flat direction $N = 2$ prepotential and hence break $N = 2$ supersymmetry
    - ∗ With decoupling theorem, integrating out heavy modes $\Longrightarrow N = 1$ $SU(2)$ SUSY YM theory

- **Two vacua in** $N = 1$ $SU(2)$ **YM theory are just two singular points** $u = \pm\Lambda^2$ **on quantum moduli space of** $N = 2$ $SU(2)$ **SUSY YM theory in Coulomb phase**

$\Longrightarrow$

3. **Quantitative explanation to confinement and mass gap in $N = 1$ SUSY YM theory** (**Seiberg & Witten, 1994**)

   - From the viewpoint that $N = 1$ $SU(2)$ **SUSY YM theory is considered as the low-energy effective theory for** $N = 2$ $SU(2)$ **YM theory in Coulomb phase** $\Longrightarrow$

     The mass gap in low-energy $N = 1$ SUSY YM theory is equivalent to the fact that massless $U(1)$ photon in the Coulomb phase of $N = 2$ low-energy YM theory must become massive by Higgs mechanism at $u = \pm\Lambda^2$

   - **At $u = \pm\Lambda^2$, the $N = 2$ SUSY YM theory is at strong coupling region, but is equivalently described by a weakly coupled magnetic dual description — $N = 2$ $U(1)$ SUSY gauge theory with monopole or dyon as hypermultiplet ($N = 2$ matter)**

   - At $u = \Lambda^2$, **the field which condensate, break magnetic $U(1)$ gauge stymmetry and carry the Higgs mechanism out is massless monopole**

     $\Longrightarrow$ **Dula Higgs mechanism**

   - **This is exactly dual superconductivity mechanism quantitatively realized in the strong coupling region of the Coulomb phase of the $N = 2$ SUSY $SU(2)$ YM theory**

$\Longrightarrow$ **confinement and mass gap in low-energy** $SU(2)$ $N=1$ **SUSY YM theory**

- **Field theory description**

Only consider **superpotential of dual magnetic theory** near $u = \Lambda^2$

$$W = \sqrt{2} A_D M \widetilde{M} + m \mathrm{Tr} \Phi^2 \tag{85}$$

M, $\widetilde{M}$ = monopole hypermultiplet

$\Longrightarrow$ Perturbation on $N=2$ vacuum moduli space

$$\begin{aligned} W &= \sqrt{2} a_D M \widetilde{M} + m u(a_D) \\ &\Longrightarrow \frac{\partial W}{\partial a_D} = \frac{\partial W}{\partial M} = \frac{\partial W}{\partial \widetilde{M}} = 0 \\ &\sqrt{2} M \widetilde{M} + m \frac{du}{da_D} = 0, \\ &a_D M = a_D \widetilde{M} = 0 \end{aligned} \tag{86}$$

$\Longrightarrow$ F-flat directions= new quantum moduli space
There are two cases:

(a) If $m=0 \Longrightarrow$

$$\langle M \rangle = \langle \widetilde{M} \rangle = 0, \quad \text{arbitrary } a_D \tag{87}$$

$\Longrightarrow$ $N=2$ quantum moduli space

(b) However, if $m \neq 0 \Longrightarrow$

$$\langle a_D \rangle = 0 \;\Leftrightarrow\; u = \Lambda^2$$

$$\langle M \rangle = \langle \widetilde{M} \rangle = \left(-\frac{m u'(0)}{\sqrt{2}} \right)^{1/2},$$

$$u'(0) = \left. \frac{du(a_D)}{da_D} \right|_{a_D=0} \tag{88}$$

$\Longrightarrow N = 1$ vacuum at $u = \Lambda^2$;
Monopole M and $\widetilde{M}$ condensation ;
$N = 2$ dual $U(1)$ supermultiplet A_D get mass with dual magnetic Higgs mechanism

6 Summary

Development of non-perturbative quantum field theory in the aspect relevant to Seiberg-Witten theory

- **Theoretical model:**

  Landau-Ginzburg theory $\longrightarrow$ **Abelian Higgs** theory $\longrightarrow$ **Georgi-Glashow** model $\longrightarrow$ **Seiberg-Witten** theory

- **Theoretical promotion:**

  Non-relativistic theory $\longrightarrow$ **Abelian relativistic** generalization $\longrightarrow$ **Non-Abelian relativistic** generalization $\longrightarrow$ **Supersymmetric** generalization

- **Physical object:**

  Abrikosov vertex solution $\longrightarrow$ **Nielsen-Olesen vertex** solution $\longrightarrow$ **'t Hooft-Polyakov magnetic monopole** $\longrightarrow$ Super-magnetic monopole (=**supermultiplet of BPS magnetic monopole**)

Significance of Seiberg and Witten's work on non-perturbative $N = 2$ $SU(2)$ SUSY gauge theory

1. **For the first time**, an exact non-perturbative solution in **four-dimensional** gauge theory has been discovered but supersymmetry must be present

2. **Electric-magnetic duality plays a crucial role** in looking for solution and provide a physical explanation; This is **the first time** that duality has application in theoretical energy physics

3. Based on exact solution, quantitative and explicit mechanisms for confinement and chiral symmetry breaking have been revealed **for the first time** from the fundamental theory

4. To some extent it has **tolled the bell** for the second superstring revolution starting from 1997

5. It has **brought gospel** to modern differential geometry and made the computation on Donaldson invariant **" at least a thousand times easier "** (from eminent mathematician C.H. Taubes)

   - **Donaldson invariant** — a polynomial topological invariant distinguishing delicate differential structure of a smooth four-dimensional manifold,

6. It has brought about " **duality era** " of quantum field theory

   - Before 1990s, it is " **symmetry era** ", algebraic tool is **continuous Lie group and Lie algebra**;

   - After that , we come to " **duality era** ", application of large **discrete group** which change coupling constant in 4-D and higher dimensional field theory becomes popular

   - **Duality in SUSY gauge theory is a field theory realization of non-perturnative phenomena of string**

theory e.g., coupling τ is related to dilaton

$\Longrightarrow$ **go to string theory** for a clear understanding on the origin of duality

7. For practical physical application, the challenge is going to **non-supersymmetry case** and tackling non-pertubative dynamics of ordinary QCD !!