Path Differentials

- Characterizing a lens
- Definition of path differentials
- Point eikonal
- Reciprocity theorem
- Application
Characterizing a lens

From plane $z=0$, ray is specified by position $(x,y)$ and direction $(L, M, N)$.

$$N = \sqrt{1 - L^2 - M^2}$$

Lens can be characterized by the four parameters.
Path differentials

A and A’ are points on the ray
R and R’ are Q and Q’ projected on P and P’

Optical path difference:

\[ QQ' - PP' = QAA'Q' - PAA'P \]

\[ = (A'Q' - A'P') - (PA - QA) \]

\[ \sim P'R' - PR \]

(triangles QAR and Q’A’R’ are small)
Because of projection,

$$\overline{QQ'} - \overline{PP'} \sim \overline{P'R'} - \overline{PR}$$

$$= n' dr' \cos \theta' - ndr \cos \theta$$

**Unit vector in propagation**

$s = (L, M, N)$ and $s' = (L', M', N')$

$$\overline{QQ'} - \overline{PP'} \sim n' s' \cdot dr' - ns \cdot dr$$

$$= n'(L' dx' + M' dy' + N' dz') - n(Ldx + Mdy + Ndz)$$

**Differential form of Hamilton’s point characteristics $E$**

$$dE(x, y, z; x', y', z') = n'(L' dx' + M' dy' + N' dz')$$

$$- n(Ldx + Mdy + Ndz)$$
Condition of point characteristics

\[ dE(x, y, z; x', y', z') = n'(L'dx'+M'dy'+N'dz') - n(Ldx + Mdy + Ndz) \]

\[
\begin{align*}
\frac{\partial E}{\partial x} &= -nL, & \frac{\partial E}{\partial x'} &= n'L', \\
\frac{\partial E}{\partial y} &= -nM, & \frac{\partial E}{\partial y'} &= n'M', \\
\frac{\partial E}{\partial z} &= -nN, & \frac{\partial E}{\partial z'} &= n'N'.
\end{align*}
\]

\[
\left(\frac{\partial E}{\partial x}\right)^2 + \left(\frac{\partial E}{\partial y}\right)^2 + \left(\frac{\partial E}{\partial z}\right)^2 = n^2,
\]

\[
\left(\frac{\partial E}{\partial x'}\right)^2 + \left(\frac{\partial E}{\partial y'}\right)^2 + \left(\frac{\partial E}{\partial z'}\right)^2 = n'^2.
\]
The Point Eikonal

- Hamilton’s point characteristics is awkward to work with.
- Bruns’ point eikonal $S$ is simpler

$$S(x, y; x', y') = E(x, y, 0; x', y', 0)$$

$$dS(x, y; x', y') = n'(L'dx' + M'dy') - n(Ldx + Mdy)$$

$$\frac{\partial S}{\partial x} = -nL,$$  $$\frac{\partial S}{\partial x'} = n'L',$$

$$\frac{\partial S}{\partial y} = -nM,$$  $$\frac{\partial S}{\partial y'} = n'M'.$$
**Reciprocity theorem**

\[
\begin{align*}
\frac{\partial S}{\partial x} &= -nL, & \frac{\partial S}{\partial x'} &= n'L', \\
\frac{\partial S}{\partial y} &= -nM, & \frac{\partial S}{\partial y'} &= n'M'.
\end{align*}
\]

\[
\begin{align*}
\frac{\partial^2 S}{\partial x \partial x'} &= n'\left(\frac{\partial L'}{\partial x}\right) \bigg|_{x'} = \frac{\partial^2 S}{\partial x' \partial x} = -n\left(\frac{\partial L}{\partial x'}\right) \bigg|_{x} \\
\frac{\partial^2 S}{\partial y \partial y'} &= n'\left(\frac{\partial M'}{\partial y}\right) \bigg|_{y'} = \frac{\partial^2 S}{\partial y' \partial y} = -n\left(\frac{\partial L}{\partial y'}\right) \bigg|_{y}
\end{align*}
\]
\[ n'(\frac{\partial L'}{\partial x'}) \bigg|_{x'} = -n(\frac{\partial L}{\partial x'}) \bigg|_{x} \]

This is a good way to check your results or data.
**Application of eikonal theory: symmetric camera lenses**

**Procedures:**
1. Determine eikonal $S$
2. Derive more properties

Eikonal can be determined from

\[
\frac{\partial S}{\partial x} = -nL', \quad \frac{\partial S}{\partial x'} = n'L',
\frac{\partial S}{\partial y} = -nM', \quad \frac{\partial S}{\partial y'} = n'M'.
\]

We need to think carefully about the system setup (ray propagation) to solve the eikonal $S$ from above set of equations.
Tips:
(1) We can choose the z and z’ axes along the axis of rotation;
(2) We can place the reference planes symmetrically in the front focal plane and back focal plane;
(3) We can restrict our discussion to rays in the plane of drawing, such that y and y’ are equal to zero.

Next: Starts to think about rays

A source point at infinity emits a beam of light which is parallel by the time it reaches the lens. The beam is focused on the back focal plane with a height \( x' \).

So, \( L \) and \( x' \) determine each other uniquely

\[
\frac{\partial S}{\partial x} = -nL = -nF(x')
\]

\[
S(x; x') = -xF(x') + K(x')
\]

\( K(x') \): an integral constant

Set \( n=1 \) for simplicity
Because of symmetry, so $S(x;x')=S(x';x)$

\[ -xF(x') + K(x') = -x'F(x) + K(x) \]

Hence, $K$ is a constant and $F(x)$ is proportional to $x$ only

\[ S(x; x') = -axx' + K \]

Next, let us to derive more result from the conditions

\[
\begin{align*}
\frac{\partial S}{\partial x} &= -nL, & \frac{\partial S}{\partial x'} &= n'L', \\
\frac{\partial S}{\partial y} &= -nM, & \frac{\partial S}{\partial y'} &= n'M'.
\end{align*}
\]

So, we have $-L=-ax'$, or $x'=L/a$. Now $L$ is the sine of the angle between the incident light and axis of the lens. Hence, $a=1/f$ where $f$ is the focal length of the lens.

\[ x' = f \sin \theta \]
Problem 3-1

- A point eikonal is given by $S(x,x')=0.5 \ ax^2+bxx'+0.5cx'^2$. Show that it is impossible for the constant $b$ to be zero.
- Give a physical example of the meaning $b$ to indicate why it can’t be zero.
- Due March 19, 2002
Problem 3-2

- To describe the propagation of rays in vacuum we can use two reference planes \((x,y)\) and \((x',y')\) separated by a distance \(a\). Show that the point ekional for this case is given by

\[
S(x, y; x', y') = \sqrt{a^2 + (x'-x)^2 + (y'-y)^2}
\]

also show that the rays follow

\[
\begin{align*}
\frac{\partial S}{\partial x} &= -nL, & \frac{\partial S}{\partial x'} &= n'L', \\
\frac{\partial S}{\partial y} &= -nM, & \frac{\partial S}{\partial y'} &= n'M'.
\end{align*}
\]

- Due March 19, 2002
You should know

- How to derive Burns’ simplified point eikonal for several typical systems, e.g. symmetrical lenses
- How to determine the trajectory of rays based on eikonal