Fermat’s principle

- Optical path
- A simple derivation of lens maker equation, …
- On perfect imaging
Geometrical Optics: Fermat’s principle

Light travels from a point \( P \) to a point \( P' \)

Time \( t(P, P') \)

\[
 t(P, P') = \int_{P}^{P'} \frac{ds}{c/n} = \frac{1}{c} \int_{P}^{P'} n ds
\]

\( ds \): the length of the line elements along the ray

A real optical path should not change even there is a virtual displacement, this means that the optical path is stationary as the virtual variation occurred.
Example 1: refraction by a curved surface

\[ s = r - \sqrt{r^2 - h^2} \]

\( x, h, x' \) are small
\( r, p, q \) are large

(a small-angle approximation)

Optical path

\[ PP' = E(h) = n\sqrt{(p + s)^2 + (h - x)^2} + n'\sqrt{(q - s)^2 + (h - x')^2} \]

If we slightly change \( h \), then \( E(h) \) should be no change such that the optical path is a true trace of light ray.
perturbation

\[ s = r - \sqrt{r^2 - h^2} \sim \frac{1}{2} \frac{h^2}{r} \]

\[ \overline{PP'} = E(h) = n \sqrt{(p + s)^2 + (h - x)^2} + n' \sqrt{(q - s)^2 + (h - x')^2} \]

\[ \sim A + Bh + Ch^2 \]

\[ A = np + n'q + (nx2)/2p + (n'x'2)/2q \]

\[ B = -nx/p - n'x'/q \]

\[ C = [n/p + n'/q - (n' - n)/r]/2 \]

Stationary virtual displacement

\[ E(h + \delta h) \rightarrow A + Bh + Ch^2 + (B + 2Ch)\delta h \]

\[ (B + 2Ch) = 0 \]

\[ \frac{n' x' - h}{q} = n \frac{h - x}{p} - h \frac{n' - n}{r} \]
This is also the paraxial imaging of properties of a general lens system.
Example 2: **Cartesian surfaces** (a surface that yields perfect imaging between a single pair of points)

\[ P' \text{ is the perfect image of } P: \text{ all rays connecting } P \text{ and } P' \text{ have the same optical path} \]

\[ P' = E(h) = n\sqrt{(p-s)^2 + h^2} + n'\sqrt{(q-s)^2 + h^2} \]

So, by the ray passed \( h=0 \), we obtained the value of optical path and then, we have the equation for perfect imaging:

\[ n\sqrt{(p-s)^2 + h^2} + n'\sqrt{(q-s)^2 + h^2} = np + n'q \]

This provides us the shape equation of lens.
If \( np + n'q = 0 \), we have

\[
n \sqrt{(p - s)^2 + h^2} + n' \sqrt{(q - s)^2 + h^2} = 0
\]


\[
h^2 + 2 \frac{np}{n + n'} s + s^2 = 0
\]

A circle!!

After rotation, a sphere with radius: \( np/(n+n') \)

A spherical object surface that is imaged perfectly into a spherical image surface.
Problem 2-1

- Show that the Cartesian surface for an object at infinity is of the second degree.
- Due March 12, 2002
Problem 2-2

- Show that a Cartesian mirror is always of the second degree. When does the mirror degenerate into a plane mirror, and when into a sphere.

- Due March 12, 2002
You should know

- How to deduce more properties based on Fermat’s principle and optical path
- How to deduce the condition of perfect imaging with Fermat’s principle